Geometric Conservation Law and Finite Element Methods for 3D unsteady Simulations of Incompressible Flow

S. Etienne, A. Garon, D. Pelletier
École Polytechnique de Montréal, C.P. 6079, succ. Centre-ville, Montréal, Québec, H3C 3A7, Canada

This paper takes a fresh look at the Geometric Conservation Law (GCL) from the perspective of Arbitrary Lagrangian Eulerian (ALE) finite element methods for solving 3-D incompressible viscous flows problems on deforming domain. GCL compliance is traditionally interpreted as a consistency criterion for applying an unsteady flow solution algorithm to simulate exactly a uniform flow on a deforming domain. We introduce an additional requirement: the time integrator must maintain its fixed mesh accuracy when applied to deforming meshes. We show how a fixed mesh unsteady FEM using high order time integrator (up to fifth order in time) can be transposed to solve problems on deforming meshes and preserve its fixed mesh high order temporal accuracy. An appropriate construction of the divergence of the mesh velocity guarantees GCL compliance while a separate construction of the mesh velocity itself allows the time-integrator to deliver its fixed mesh high order temporal accuracy on deforming domains. Analytical error analysis of problems with closed form solutions provides insight on the behavior of the time integrators. We present thorough time-step and grid refinement studies for simple problems with closed form solutions and for a manufactured solution with a non-trivial flow on a deforming mesh. In all cases studied, the proposed reconstructions of the mesh velocity and its divergence lead to optimal time accuracy on deforming grids.

I. Introduction

Arbitrary Lagrangian Eulerian (ALE) formulations are very popular for solving differential equations on deforming domains. In this approach the Partial Differential Equations (PDE’s), here the unsteady Navier-Stokes equations, are expressed with respect to a reference fixed configuration. A so-called ALE mapping associates at each time \( t \) a point in the current computational domain \( \Omega(t) \) to a point in the reference domain \( \Omega(0) \). This ensures that the solution evolves along trajectories that are contained in the computational domain at all times. When discussing ALE formulations, several issues are raised by different authors. What is the regularity of the mapping required to ensure that the problem is well posed? How is the mapping defined, represented and discretized? What is the effect of the mesh velocity on the accuracy on the time integrator? What is the level of stability of the resulting algorithm? The bulk of these issues is usually discussed in the context of the GCL or Geometric Conservation Law.\(^1\)

This paper takes a fresh look at the GCL from the somewhat restricted perspective of Finite Element Methods for incompressible flows, using the usual definitions of the GCL and its discrete counterpart. However these definitions do not address the issues raised in the previous paragraph about the interactions between the mesh motion, its discretization and that of space and time discretization schemes, especially its time accuracy. As we will show, a poor choice of a single component of the ALE formulation results in a lower temporal accuracy on deforming domains than that observed on fixed domains.

**Literature review**

The GCL has been the subject of many investigations and has generated several controversial opinions. It is generally accepted that if an ALE formulation preserves a uniform flow on a deforming grid, then

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this constitutes either a definition of the GCL or a GCL compliance test for the numerical scheme. For instance Shyy et al. demonstrated that without explicitly enforcing the GCL, O(1) errors could be induced in the computations due solely to grid movement effects. Thus, it appears that GCL compliance may have a significant effect on the performance of ALE schemes.

The Space-Time Finite Element Method (STFEM), an alternative to ALE Methods, has received much attention because the GCL issue disappears completely. However this comes at the cost of doubling the number of unknowns at each time-step for a second-order time-stepping scheme. Thus there remains much interest in ALE schemes because of their lower computational expense. STFEM have been developed by many authors such as N’dri et al. for Computational Fluid Dynamics, Li et al. for Structural Dynamics, and Hansbo et al., Stein et al. for Fluid-Structure Interaction problems.

GCL compliance, or how the ALE mesh velocity is computed, often appears as a necessary condition for consistency of an ALE numerical scheme. Lacroix et al. showed that many simple techniques work well for unidirectional mesh motions. However, many schemes reveal their limitations as soon as arbitrary 2D mesh motions are considered. Lesoinne & Farhat state that compliance can be tested by verifying satisfaction of the discrete analogue of the GCL. This amounts to checking that a uniform flow is not polluted by approximation errors caused by mesh motion.

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Method</th>
<th>Form.</th>
<th>Order</th>
<th>Grid Conv.</th>
<th>Stability study</th>
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<td>Δx³, Δt²</td>
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Table 1. Methods (FE: Finite Element, FD: Finite Difference, FV: Finite Volume), formulation of PDE’s (C: Conservative, NC: Non-Conservative), discretization order, Occurrence of Grid convergence and Stability study and Incompressible flows study.

Table 1 shows a compilation of many contributions and approaches to the GCL. All but one discuss time integrators of order 1 or 2. Mavriplis discusses the construction of GCL compliant Finite Volume formulations for time-stepping schemes of order higher than 2. Thus, there is a definite need to investigate
how to develop GCL compliant ALE FEM with high order temporal accuracy. The fifth column in table 1 indicates the order of convergence in space and time of the schemes used in each paper. The next to last column of the table indicates whether or not systematic time step refinement studies were performed by the authors. Several authors show results with such complex geometries or flow physics that it most likely masks the effects of GCL compliance. In such situations, we consider that convergence studies are inconclusive even when they have been performed. The last column indicates if a theoretical stability analysis has been done.

Previous studies used various names and acronyms to refer to different aspects of the GCL. The concept of GCL was first introduced in 1961 by Trulio et al. and later generalized by Thomas et al. for the Finite Volume Method (FVM). Different approaches have been considered. As observed by Demirzic et al., the true significance of the Space Conservation Law (SCL) or GCL, was probably not recognized initially because most studies were performed with one-dimensional mesh motions for which it is much easier to satisfy the GCL, than on grids subjected to arbitrary motion. This is why many highly successful 1D schemes failed when generalized to two or three dimensions. Förster et al. showed that a simple formula can be used to compute the mesh velocity for 1D grid motions. However, since they do not show results for more general 2D grid motions, it cannot be inferred that their formulation satisfies the GCL in 2D or 3D.

Much of initial work on the GCL was performed using Finite Volume Methods. While the FVM and FEM often lead to identical schemes in 1D, multi-dimensional discretizations of fluid flow equations result in different computational stencils. Hence, one can legitimately expect that an effective finite volume ALE technique might not apply directly to finite element discretizations without some adjustments.

Zhang et al. discussed previous studies and their limitations with regards to general purpose application. They note that a numerical scheme is said to satisfy the Volume Conservation Law (VCL) if it guarantees satisfaction of both the continuity equation and the SCL for a uniform flow. They stress that satisfying these two laws is required to achieve GCL compliance.

**DGCL and temporal accuracy**

The Discrete Geometric Conservation Law (DGCL) of Farhat et al. provides significant insight on the accuracy of the time-stepping scheme. For a $p^{th}$-order time-accurate scheme ($p \geq 1$) on a fixed mesh, satisfying the discrete geometric conservation law to $p^{th}$-order accuracy is a sufficient condition for this scheme to be at least first-order time-accurate on a moving mesh, meaning that in practice the time accuracy of a DGCL compliant scheme might be reduced from $p^{th}$ order on a fixed mesh to as low as first order on a deforming mesh, a catastrophic loss of temporal accuracy. Furthermore, the higher the order of a time-stepping scheme on a fixed grid, the more important it becomes for this scheme to satisfy the DGCL for successful applications to flows on a moving grid. Furthermore, unless restricted by numerical stability conditions, a numerical method satisfying the DGCL will generally allow a much larger computational time-step than its counterpart violating the DGCL.

Mavriplis et al. as well as Geuzaine et al. showed that it is possible to construct an ALE formulation which preserves the fixed mesh temporal accuracy of the time-stepping scheme on a deforming mesh without satisfying the DGCL. Moreover, Mavriplis also states that the DGCL is unrelated to the time-accuracy! Geuzaine et al. have proven that the DGCL is neither a necessary nor a sufficient condition to preserve time-accuracy.

**DGCL and stability**

The effect of GCL on stability of ALE Methods is controversial. This question is still open for debate as evidenced by the opposing conclusions of Farhat et al. and Boffi et al. In the paper by Farhat the DGCL appears as a necessary and sufficient condition for stability of schemes with up to 2nd order temporal accuracy, while Boffi et al. and Formaggia and Nobile showed that it is neither necessary nor sufficient for stability except for the 1st order implicit backward Euler scheme. They conclude that GCL compliance is likely to improve accuracy of the numerical scheme and enhance its stability in some cases.

Mavriplis compared non DGCL-compliant to DGCL-compliant schemes of the same temporal accuracy. The compliant schemes appear to be free from oscillations that are unavoidable with the non DGCL-compliant ones. He attributed these oscillations to the lack of conservation induced by violation of the GCL. In their work, Farhat et al. showed too that schemes violating the DGCL while generating accurate solutions, were polluted by bounded oscillations whereas those satisfying the DGCL didn’t exhibit spurious oscillations. Numerical consistency requires satisfaction of the Geometric Conservation Law (GCL) to avoid spurious solutions.

Kamakoti et al. present an unusual approach. The flow field is advanced in time to $t^{n+1}$ using the
implicit Euler scheme while a GCL compliant mesh velocity field is obtained at $t^{n+1}$ with variants of first or second order implicit time-integrators. They found that the best results are obtained when the first order implicit Euler scheme is applied to both the flow and mesh velocity.

**The present work**

We take a different approach: given a spatial discretization and a time-integrator, we seek a method for evaluating the divergence of the mesh velocity to satisfy the GCL, and a second method to compute the mesh velocity so as to ensure that high temporal accuracy observed on fixed meshes is maintained on moving meshes. In a sense we separate the computation of the mesh velocity and that of its divergence as if they arose from two distinct mesh velocity fields. We believe that this approach offers several advantages. Code development and implementation are simplified as the time integrator routine remains unchanged. In fact, the same routine is used in 2D and 3D and for fixed and deforming meshes. Modifications are confined to a handful of routines for evaluation of the weak form residuals, mesh velocity divergence, and mesh velocity itself. Thus development time is reduced. Code maintenance is simplified since there is no need to maintain separate integrators for fixed mesh and deforming mesh simulations nor is there any requirements to use different integrators in 2D/3D simulations. Finally, because identical integrators are used in all cases, analysis of results, and diagnosis of odd simulation results are also simplified.

Furthermore, incompressible flows present an additional difficulty. Because pressure acts as the Lagrange multiplier enforcing the incompressibility constraint (continuity equation), achieving higher order time accuracy for pressure is more difficult in incompressible flows than in compressible flows. In fact, for time-stepping schemes of order greater than 2 for the velocity, incompressibility is achieved at the cost of a reduction of the temporal accuracy in pressure. In this study, we consider the Crank-Nicholson ($2^{nd}$ order for velocity and pressure) and a family of fully and singly-diagonally implicit Runge-Kutta schemes that are $1^{st}$ to $5^{th}$ order time accurate for velocity and $1^{st}$ to $3^{rd}$ order time accurate for pressure.

To shed some light we propose testing the GCL at 3 levels of compliance:

1. **GCL level 1**: an Arbitrary Lagrangian Eulerian (ALE) formulation satisfying the GCL must produce the exact solution to the no-flow test on arbitrary deforming grids (see\cite{16}).

2. **GCL level 2**: an ALE formulation satisfying the GCL must produce the exact solution of a uniform flow on moving grids, this is the so-called Discrete GCL (DGCL) by Farhat *et al.*\cite{19}

3. **GCL level 3**: for a properly designed ALE formulation satisfying the GCL, the time-stepping scheme must exhibit the same convergence rate on deforming and fixed grids.\cite{19, 26}

The first two levels are commonly found in the literature. The third one is a powerful test of time integrators, especially high order ones.

We show through carefully chosen examples that satisfying the constant solution test on a moving grid, *i.e.* the usual standard GCL compliance test, does not guarantee that the high order temporal accuracy observed on fixed meshes will also occur on moving meshes.

The paper is organized as follows. Section II describes the governing equations for laminar flows of incompressible fluids on deforming domains. This is followed in section III by a brief overview of the finite element method used to solve the equations while section IV is a discussion of the conservative and non-conservative weak forms of the ALE problems. Section V provides a general framework useful to compare time-integrators. Section VI discusses methods to evaluate the mesh velocity divergence so as to satisfy the GCL. The separate construction of the mesh velocity itself to achieve high order temporal accuracy is the subject of section VII. Section VIII presents results from careful numerical tests and analytical error analysis. Section IX presents a discussion of the results and conclusions.

**II. Governing Equations**

The flow of an incompressible fluid is described by the continuity and momentum equations\cite{30} written in *convective* (non-conservative) form

\[
\nabla \cdot \mathbf{u} = 0 \tag{1}
\]

\[
\rho \mathbf{u}_t + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \cdot \mathbf{\sigma} + \mathbf{f} \tag{2}
\]
In an arbitrary time-dependent coordinate system, the momentum equations (2) are written, in non-conservative form as

\[ \rho \frac{du}{dt} + \rho ([u - v] \cdot \nabla)u = \nabla \cdot \sigma + f \]  

(3)

where \( v \) is the velocity of the moving reference frame, \( \rho \) the fluid density, \( u \) the fluid velocity, \( \sigma \) the total fluid stress tensor (pressure and viscous forces), and \( f \) a body force. Eq. (1) and (2) are expressed in an Eulerian frame of reference while Eq. (3) is expressed in an Arbitrary Lagrangian Eulerian (ALE) coordinate system. Details of its development can be found in.\textsuperscript{14} Assuming that the fluid is Newtonian, its constitutive equation is given by

\[ \sigma = \tau - pI \]  

with \( \tau = \mu [\nabla u + (\nabla u)^T] \)

where \( \mu \) is the dynamic viscosity and \( p \) is the fluid pressure. The flow equations are closed with the following boundary conditions,

\[ \sigma \cdot n = \bar{t} \text{ on } \Gamma_N \]  

(4)

\[ u = \bar{u} \text{ on } \Gamma_D \]

where \( \Gamma_N \) denotes a boundary where Neumann conditions are applied in the form of prescribed surface forces (tractions) \( \bar{t} \), and \( \Gamma_D \) corresponds to a Dirichlet boundary on which the velocity, \( \bar{u} \), is imposed.

### III. Solution Strategy

We have chosen a monolithic solution strategy coupling all degrees of freedom: velocities, pressure, along with pseudo-solid displacements sued to define the domain deformation. In this approach all equations are treated implicitly in the time integration scheme. This methodology is applicable not only for flows on deforming meshes but also to fully implicit monolithic treatment of unsteady Fluid-Structure Interactions. Linearization of the flow and mesh equations must account for all implicit dependencies to ensure quadratic convergence of Newton’s method.\textsuperscript{31} These steps are implemented simply and in a straight forward manner through the use of numerical Jacobians. This approach is very robust and applicable to a broad spectrum of problems. Note, however, that the details of the finite element flow solver have no effect on GCL compliance.

The fluid velocity and displacement fields are discretized using 6-node quadratic elements in 2-D and 10-node tetrahedra in 3-D. Fluid pressure is discretized by piecewise linear continuous functions. This discretization proves to be useful in our study because all linear and quadratic flows are captured exactly (Couette and Poiseuille for instance) so that the spatial error is zero leaving time discretization as the sole contributor to error. In such cases, diagnosis of the temporal order of accuracy becomes easy. The resulting sparse matrix system is solved using the PARDISO software.\textsuperscript{32,33}

### IV. Integral Formulation and the Geometric Conservation Law

We focus on the conservative weak form of the Navier-Stokes equations.

\[
\frac{d}{dt} \int_{\Omega(t)} \mathbf{w} : \rho \mathbf{u} d\Omega - \int_{\Gamma(t)} \mathbf{w} \cdot (\nabla \cdot \mathbf{v}) \rho d\Omega \\
+ \int_{\Omega(t)} \mathbf{w} \cdot \{ \rho ([u - v] \cdot \nabla)u + \nabla \mathbf{w} : \sigma \} d\Omega = \int_{\Gamma_N(t)} \mathbf{w} \cdot (\sigma \cdot n) d\Gamma
\]  

(5)

Note that we assume that the transformation \( T^{(t)} \) from \( \Omega(0) \) to \( \Omega(t) \) to be regular at all times i.e. of class \( C^{(1)} \), univalent and such that \( J(t)T^{(t)} \neq 0 \) on \( \Omega(0) \). \( J(t) \) is the determinant of the Jacobian of the transformation from \( \Omega(0) \), the reference domain, to \( \Omega(t) \), the domain.

For the formulation given in Eq. (5) the time derivative must account for the time dependence of both the integrand and the domain of integration. For this formulation, the mesh velocity divergence must be evaluated in such a way that the GCL is satisfied. This is not the case for the non-conservative formulation for which the time derivative is under the sum sign and for which the mesh velocity divergence does not appear. We have shown previously\textsuperscript{34} that, while not inconsistent, this last formulation doesn’t provide GCL compliance at level 3. In fact, only first order temporal accuracy is observed on moving meshes even for those time integrators which have shown high order temporal accuracy on fixed meshes (see Etienne et al.\textsuperscript{34}).
V. Time-stepping schemes considered

We study six time integration schemes to assess their interactions with the weak form (5) and their effects on the predictions. We consider a variety of Runge-Kutta schemes ranging from the first order implicit Euler scheme to the 5\textsuperscript{th} order Radau IIA scheme. All these schemes are fully implicit and their temporal accuracy on fixed meshes has been assessed theoretically and numerically by Hairer \textit{et al.}\textsuperscript{35} Key results are summarized in table 2.

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<th>Scheme</th>
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Table 2. Fixed mesh temporal accuracy of time integrators

To better appreciate the effects of the time-stepping scheme, we first illustrate their use on the sample ODE \( y' = \phi(t, y) \). A general RK scheme applied to this equation reads

\[
y_s^{(n+c_i)} = y^{(n)} + \Delta t \sum_{j=1}^{s} a_{ij} \phi(t^{(n+c_j)}, y_s^{(n+c_j)}), \text{ for } i = 1, \ldots, s
\]

\[
y^{(n+1)} = y^{(n)} + \Delta t \sum_{j=1}^{s} b_j \phi(t^{(n+c_j)}, y_s^{(n+c_j)})
\]

where \( t^{(n+c_j)} \) means \( t^{(n)} + c_j \Delta t \). This system is summarized compactly in the form of a general Butcher Tableau

\[
\begin{array}{cccc}
  c_1 & a_{11} & \ldots & a_{1s} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_s & a_{s1} & \ldots & a_{ss} \\
\end{array}
\begin{array}{c}
  b_1 \\
  \vdots \\
  b_s
\end{array}
\]

Following,\textsuperscript{35} for all the schemes studied here, we have \( a_{si} = b_i \), for \( i = 1 \ldots s \) to ensure L-stability so that the projection step defined by Eq. (7) can be skipped. Note also that the RK schemes are constructed to have the following property \( c_i = \sum_{k=1}^{s} a_{ik} \), for \( i = 1 \ldots s \).

The Crank-Nicholson time-stepping scheme is of Lobatto IIIa type and is 2\textsuperscript{nd} order accurate for both velocity and pressure. Its Butcher Tableau is

\[
\begin{array}{cc}
  0 & 0 \\
  1 & 1/2 \\
\end{array}
\begin{array}{cc}
  1/2 \\
  1/2
\end{array}
\]

The first row corresponds to an explicit step so that we can write the 2-step RK scheme in a single line as

\[
y^{(n+1)} = y^{(n)} + \frac{\Delta t}{2} \phi(y^{(n)}, y^{(n)}) + \frac{\Delta t}{2} \phi(y^{(n+1)} + \Delta t, y^{(n+1)})
\]

According to Hairer \textit{et al.},\textsuperscript{35} because the pressure is a Lagrange multiplier for incompressibility, the Navier-Stokes equations for incompressible flows are a system of index 2 Differential Algebraic Equations which implies a lower time accuracy for the pressure. The Butcher Tableaus of these two schemes are as follows
which translates to

\[ y^{(n+1/3)} = y^{(n)} + \frac{5\Delta t}{12} \phi(t^{(n+1/3)}, y^{(n+1/3)}) - \frac{\Delta t}{12} \phi(t^{(n+1)}, y^{(n+1)}) \]  
(9)

\[ y^{(n+1)} = y^{(n)} + \frac{3\Delta t}{4} \phi(t^{(n+1/3)}, y^{(n+1/3)}) + \frac{\Delta t}{4} \phi(t^{(n+1)}, y^{(n+1)}) \]  
(10)

and means that the two implicit steps are coupled. Thus, doubling the number of unknowns when advancing from \( t^{(n)} \) to \( t^{(n+1)} \) compared to CN. Finally, the Radau IIA5 can be summarized as:

<table>
<thead>
<tr>
<th></th>
<th>(4-(\sqrt{6}))/10</th>
<th>(88-7(\sqrt{6}))/360</th>
<th>(296-169(\sqrt{6}))/1800</th>
<th>(-2+3(\sqrt{6}))/225</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4+(\sqrt{6}))/10</td>
<td>(296+169(\sqrt{6}))/1800</td>
<td>(88+7(\sqrt{6}))/360</td>
<td>(-2-3(\sqrt{6}))/225</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(16-(\sqrt{6}))/36</td>
<td>(16+(\sqrt{6}))/36</td>
<td>1/9</td>
<td></td>
</tr>
</tbody>
</table>

IRK5

which translates to

\[ y^{(n+c_1)} = y^{(n)} + \sum_{j=1}^{3} a_{1j} \Delta t \phi(t^{(n+c_j)}, y^{(n+c_j)}) \]  
(11)

\[ y^{(n+c_2)} = y^{(n)} + \sum_{j=1}^{3} a_{2j} \Delta t \phi(t^{(n+c_j)}, y^{(n+c_j)}) \]  
(12)

\[ y^{(n+c_3)} = y^{(n)} + \sum_{j=1}^{3} a_{3j} \Delta t \phi(t^{(n+c_j)}, y^{(n+c_j)}) \]  
(13)

which leads to three coupled implicit steps thus tripling the number of unknowns compared to CN.

Finally, the three other schemes possess the following Butcher tableaus:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRK1</td>
<td>1</td>
</tr>
<tr>
<td>\gamma</td>
<td>\gamma 0</td>
</tr>
<tr>
<td>1</td>
<td>1-\gamma \gamma</td>
</tr>
</tbody>
</table>

SDIRK 21

with \( \gamma = (1+\sqrt{2})/2 \), which is a solution of \( x^2 L_2(1/x) = 0 \), \( L_2 \) being the 2\(^{nd}\)-degree Laguerre Polynomial.

<table>
<thead>
<tr>
<th></th>
<th>\gamma</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>\gamma</td>
<td>\gamma 0</td>
<td></td>
</tr>
<tr>
<td>\gamma</td>
<td>\gamma 0</td>
<td></td>
</tr>
</tbody>
</table>

SDIRK 31

with \( \gamma = 0.435866521508458 \), which is a solution of \( x^3 L_3(1/x) = 0 \), \( L_3 \) being the 3\(^{rd}\)-degree Laguerre Polynomial.

VI. Evaluating the mesh velocity divergence

From here on we assume that space has been discretized by a suitable finite element method with straight-sided triangles or tetrahedra which implies that the mesh motion and the mesh velocity are piecewise linear functions of space. The fluid velocity and pressure are discretized with a suitable finite element scheme. We chose the Taylor-Hood element or P2-P1 element for which velocity is discretized with quadratic polynomials.
and pressure approximated by linear polynomials. This element is 2\textsuperscript{nd} order accurate for the velocity in $H^1$ norm and 2\textsuperscript{nd} order accurate for the pressure in $L_2$ norm.

The uniform flow test provides a rule for calculating the divergence of the mesh velocity, $(\nabla \cdot \mathbf{v})$, to satisfy the GCL for the conservative formulation Eq. (5). As this is done at the discrete level, it is referred to as the DGCL. This implies compliance at levels 1 and 2, see section IV.

Mavriplis studied two time integrators BDF2 and ESDIRK64.\textsuperscript{20} To satisfy the DGCL for each integrator he requires that the uniform flow satisfy the discretized equations. He does this for a Finite Volume formulation of compressible flows i.e. GCL compliance at levels 1 and 2. In the case of FVM, doing this defines a mesh velocity consistent with the space and time discretization schemes.

Enforcing GCL compliance on the FEM conservative formulation (5) provides a rule for evaluating the divergence of the mesh velocity which will be denoted by $(\nabla \cdot \mathbf{v}_d)$. However, it provides no rule on how to evaluate $\mathbf{v}_d$ itself, meaning that it does not define which mesh velocity $\mathbf{v}_d$ should be used in the ALE FEM. In the present work, the rule for constructing the mesh velocity depends on whether the Butcher Tableau of the scheme is invertible or not.

### A. GCL for time-stepping schemes with invertible Butcher tableaus

In this section, we consider time-stepping schemes for which Butcher tableaus are invertible. This embraces all Gauss, Radau and SDI Runge-Kutta schemes, i.e. IRK3 and 5 (see\textsuperscript{35}). We also assume, as explained earlier in section II, that the mapping between $\Omega(0)$ and $\Omega(t)$ is given by a piecewise linear representation on the elements of the mesh. The Jacobian of this transformation is locally constant on each element of the mesh. Note however that $J$ will be a function of time since the geometry of the element will change with time.

In this section we recall theoretical results from our previous work,\textsuperscript{34} in which formal proofs are valid for both 2-D and 3-D problems. However, numerical tests were carried out in 2-D only.

**Proposition VI.1** Consider a time-stepping scheme with an invertible Butcher tableau $A = [a_{ij}]_{i,j \in [1,s]^2}$ and let $B = [b_{ij}]_{i,j \in [1,s]^2} = A^{-1}$ be its inverse. Let $J^{(t)}$ be the determinant of the Jacobian transformation at time $t$ from the reference domain $\Omega(0)$ to the deforming domain $\Omega(t)$. If we restrict ourselves to straight-sided triangular or tetrahedral elements, the following mesh velocity divergence formulation is a sufficient condition for compliance of the conservative formulation to levels 1 and 2 of the GCL (no-flow and uniform flow).

$$
(\nabla \cdot \mathbf{v}_d)^{(n+c_j)} = \sum_{j=1}^{s} b_{ij} J^{(n+c_j)} - J^{(n)} \Delta t / J^{(n+c_j)}, \ j = 1...s \quad (14)
$$

### B. GCL for time-stepping schemes with non-invertible Butcher tableaus

For some popular time integrators, such as the Crank-Nicholson scheme, $(a_{ij})$ is not invertible. It is possible, however, to proceed in much the same way as in the previous section. We treat schemes which do not need a projection step. This means schemes for which the coefficients in Eq. (7) are equal to those of the last line of system (6), last row in the Butcher tableau. We exclude for brevity’s sake Lobatto IIIB scheme, which would require consideration of the projection step in the discussion.

**Proposition VI.2** Consider a time-stepping scheme which doesn’t require a projection step and for which the Butcher tableau $A = [a_{ij}]_{i,j \in [s-p+1,s] \times [1,s]}$ is not invertible but is of rank $p$ equal to the number of implicit steps, from the $s - p + 1$\textsuperscript{th} to the $s$\textsuperscript{th} step, the last $p$ rows of $A$. Let $J^{(t)}$ be the determinant of the Jacobian transformation at time $t$ from the reference domain $\Omega(0)$ to the deforming domain $\Omega(t)$. Then, if we restrict ourselves to straight-sided triangular or tetrahedral elements, there exists at least one vector $[(\nabla \cdot \mathbf{v}_d)^{(n+c_j)}]_{j \in [1,s]}$ such that

$$
\frac{J^{(n+c_j)} - J^{(n)}}{\Delta t} = \sum_{j=1}^{s} a_{ij} J^{(n+c_i)} (\nabla \cdot \mathbf{v}_d)^{(n+c_i)}, \ k = s - p + 1...s \quad (15)
$$
Then this constitutes a sufficient condition for compliance of the conservative formulation to levels 1 and 2 of the GCL (no-flow and uniform flow tests).

For example, applying Eq. (15) to the Crank-Nicholson integrator for a uniform flow leads to the following equality (equivalent to GCL compliance at level 2) which must be satisfied

$$\frac{1}{2} J^{(n+1)} (\nabla \cdot \mathbf{v}_d)^{(n+1)} + \frac{1}{2} J^{(n)} (\nabla \cdot \mathbf{v}_d)^{(n)} = \frac{J^{(n+1)} - J^{(n)}}{\Delta t}$$

(16)

As there are two unknowns ($\nabla \cdot \mathbf{v}_d$ at times $n$ and $n+1$) several solution vectors $[(\nabla \cdot \mathbf{v}_d)^{(n+j)}]_{j \in [0,1]}$ are possible. We chose

$$(\nabla \cdot \mathbf{v}_d)^{(n)} = \frac{J^{(n+1)} - J^{(n)}}{\Delta t J^{(n)}}$$

(17)

$$(\nabla \cdot \mathbf{v}_d)^{(n+1)} = \frac{J^{(n+1)} - J^{(n)}}{\Delta t J^{(n+1)}}$$

(18)

Note that other choices are possible, for instance any convex combination of

$$(\nabla \cdot \mathbf{v}_d)^{(n)} = 2a(\nabla \cdot \mathbf{v}_d)^{(n)}$$

(19)

$$(\nabla \cdot \mathbf{v}_d)^{(n+1)} = 2(1-a)(\nabla \cdot \mathbf{v}_d)^{(n+1)}$$

(20)

$a \in \mathbb{R}$ will also work. This reflects the fact that there is more freedom in the choice of the mesh velocity divergence for time-stepping schemes with non invertible Butcher tableaus.

At this point we have a general methodology, which accounts for characteristics of the time-stepping scheme, to construct the divergence of the mesh velocity to satisfy the GCL at level 2 (DGCL) on straight-sided triangles or tetrahedra. However, satisfying the DGCL does not automatically guarantee GCL compliance at level 3. See observations by Geuzaine et al.25 In fact, satisfying the DGCL does not provide any additional information about the convergence rate of the time-stepping procedure. It simply means that the time-integrator will be at least first order accurate on deforming domains. While the technique for computing ($\nabla \cdot \mathbf{v}_d$) is general, it provides no information about the proper way to compute $\mathbf{v}_d$ itself. This is the subject of the following section.

VII. Mesh velocity calculation

The objective of this section is to define a mesh velocity such that accuracy observed on fixed grids is preserved on moving grids. Most previous works assume that the mesh velocity is given and is used to evaluate both ($\nabla \cdot \mathbf{v}$) and $\mathbf{v}$ in the conservative formulation (5). The only way to achieve GCL compliance is to find a time-integrator (or design one) such that it satisfies the GCL when the given mesh velocity is used for $\mathbf{v}$ and ($\nabla \cdot \mathbf{v}$) in Eq. (5).

We take a different approach and assume that the spatial discretization and the time integrator are given, and we seek 2 formulas: one for evaluating the divergence of the mesh velocity ($\nabla \cdot \mathbf{v}_d$) to satisfy the GCL, and a second one to compute the mesh velocity itself $\mathbf{v}_m$ so as to ensure that the high order temporal accuracy observed on a fixed mesh is maintained on a deforming mesh. This amounts to using two mesh velocities $\mathbf{v}_d$ and $\mathbf{v}_m$. As shown in section VI, only ($\nabla \cdot \mathbf{v}_d$) is required to evaluate the term $\int_{\Omega(t)} \mathbf{w} \cdot (\nabla \cdot \mathbf{v}_d) \rho \, d\Omega$ and provide GCL compliance at level 2 for Eq. (5). The velocity $\mathbf{v}_d$ itself is not required. The second mesh velocity $\mathbf{v}_m$ appears only in $\int_{\Omega(t)} \mathbf{w} \cdot (\mathbf{u} - \mathbf{v}_m) \cdot \nabla \rho \, d\Omega$. Hence, ($\nabla \cdot \mathbf{v}_m$) need not be GCL compliant. Its role is to ensure that the fixed mesh temporal accuracy of the time integrator is also observed on deforming meshes.

In this approach it is clear that GCL compliance has no effect on the temporal accuracy of the time integrator. In fact, the freedom resulting from the use of two mesh velocities provides the mechanism to maintain the fixed grid temporal accuracy when the scheme applied to deforming domains. This approach offers several advantages. The same time integrator is used for fixed mesh and deforming meshes. This greatly simplifies code structure, software maintenance and modifications, use of the code, and interpretation of results. We now take a look at possible constructions of $\mathbf{v}_m$ for the three time-integrators of the previous sections. Here too the construction of $\mathbf{v}_m$ is different depending on whether the Crank-Nicholson or IRK3 and IRK5 schemes are used just as was the case for ($\nabla \cdot \mathbf{v}_d$).
A. The Crank-Nicholson scheme

We begin by taking a look at this issue for the Crank-Nicholson scheme for which the usual and accepted way of evaluating the mesh velocity is the midpoint rule

\[ v_m(x) = \frac{x^{(n+1)} - x^{(n)}}{\Delta t} \] (21)

This formula is used to compute the mesh velocity \( v_m \). However, its use to evaluate the mesh velocity divergence is restricted to 2D problems as stated by the following proposition. See Etienne et al. for a proof. For 3D computations we use formulas (19,20). This means that different formulations are used for the mesh velocity and the mesh velocity divergence in the general case to satisfy GCL compliance at level 3.

Proposition VII.1 Consider the conservative formulation Eq. (5) and piecewise linear mesh deformation in space (linear deformation of triangles or tetrahedra). Using the divergence of Eq. (21) to evaluate the divergence of the mesh velocity, \((\nabla \cdot \mathbf{v}_d) = (\nabla \cdot \mathbf{v}_m)\), will result in GCL compliance at level 1 and 2 for 2D flows only.

B. IRK Radau schemes (invertible Butcher tableaus)

Generalizing Eq. (21) to the five other time-stepping schemes yields

\[ v_{m}^{(n+c_j)}(x) = \sum_{i=1}^{s} b_{ij} \frac{x^{(n+c_j)} - x^{(n)}}{\Delta t} \] (22)

This formula may be used to compute the mesh velocity \( v_m \). However, its use to evaluate the mesh velocity divergence is restricted to 2D problems and mesh motions that are linear functions of time, a severe restriction, as stated in Etienne et al. The only remaining question is whether the convergence rate of the time stepping scheme is optimal (GCL compliance at level 3), i.e. do we achieve the same convergence rate on moving meshes as on fixed meshes. This will be answered in section VIII where we perform numerical tests on the IRK3 and IRK5 time integrators using nonlinear mesh motions.

We briefly summarize our results thus far. The conservative weak form from Eq. (5) will be GCL compliant if

1. the mesh consists of linearly deforming straight-sided triangles in 2D or tetrahedra in 3D,
2. the following definitions are used for \((\nabla \cdot \mathbf{v}_d)\) and \(v_m\) for SDIRK21, SDIRK 31, IRK1, IRK3 and IRK5

\[ (\nabla \cdot \mathbf{v}_d)^{(n+c_j)} = \sum_{i=1}^{s} [a_{ij}]^{-1} \frac{J^{(n+c_j)} - J^{(n)}}{\Delta t J^{(n+c_j)}}, j = 1...s \] (23)

\[ v_{m}^{(n+c_j)}(x) = \sum_{i=1}^{s} [a_{ij}]^{-1} \frac{x^{(n+c_j)} - x^{(n)}}{\Delta t}, j = 1...s \] (24)

or the following definitions for \((\nabla \cdot \mathbf{v}_d)\) and \(v_m\) for the Crank-Nicholson scheme

\[ (\nabla \cdot \mathbf{v}_d)^{(n+j)} = \frac{J^{(n+1)} - J^{(n)}}{\Delta t J^{(n+j)}}, j = 0,1 \] (25)

\[ v_{m}^{(n+j)}(x) = \frac{x^{(n+1)} - x^{(n)}}{\Delta t}, j = 0,1 \] (26)
Furthermore, the time integrators will exhibit optimal temporal accuracy in the sense that the fixed mesh temporal convergence rate will be observed on deforming meshes.

VIII. Analytical and numerical results

This section presents results from analytical error analysis and careful numerical tests verifying that our approach yields optimal temporal accuracy for the CN, IRK3 and IRK5 integrators. This is achieved by considering two separate classes of unsteady solutions. The first regroups solutions that are represented exactly by the spatial discretization: the uniform, Couette and Poiseuille flows. For these flows, there are no interaction or interference between the temporal and spatial discretization errors so that accuracy of the time-stepping scheme can be tested without worrying about spatial discretization artifacts.

The second kind of unsteady solutions is a manufactured solution of a very general nature that cannot be represented exactly by the finite element basis functions. We use the Method of Manufactured Solution (MMS)\(^{36}\) to construct a solution of sufficient complexity on a deforming domain to ensure that all terms in the ALE formulation are exercised. This analytical Manufactured solution makes it possible to perform time step refinement studies to draw definite conclusions about time accuracy.

A. Uniform flow test

For the uniform flow test, Eq. (5) reduces to simple algebraic equations that can be evaluated and analyzed easily. Our numerical results confirmed that no parasitic flow solution arises with either formulations, and velocities as well as pressure maintain machine accuracy for all times. In other words, both formulations satisfy the so-called uniform flow test (GCL compliance at level 1 and 2). However, it provides no means for testing GCL compliance at level 3 as levels 1 and 2 are insufficient to discriminate. Doing this requires a slightly more involved flow field as shown in the next sections. See Etienne et al.\(^{34}\) for details.

B. The linear shear flow

The linear shear flow or Couette flow is usually associated with the flow between two flat plates with one moving in its plane relative to the other. Driven by viscosity, the flow is a linear function of \(y\) (see Schlichting\(^{30}\)).

\[
\begin{align*}
  u & = 0 \\
  v & = 0 \\
  w & = y \\
  p & = \text{constant}
\end{align*}
\]

\(27) \\
\(28) \\
\(29) \\
\(30)

Even though the flow is steady one can still analyze the GCL provided the grid points inside the domain move so that the mesh deforms with time. This test exercises both the GCL and the mesh velocity \(\int_{t_0}^{t_1} \mathbf{w} \cdot \mathbf{\rho} [(u - v_m) \cdot \nabla] \mathbf{u} \neq 0\). This problem is useful because it lends itself to a complete analytical analysis of the time truncation error. Since \(\mathbf{u}\) is linear in \(y\) and \(p\) is constant, the spatial variations of the flow are represented exactly by the FE discretization. Hence the only contribution to the error is the temporal discretization.

This simplified setting is ideal to carry out a detailed time truncation error analysis of the conservative formulation, Eq. (5), with the Crank-Nicholson time integrator on arbitrary but spatially linear mesh motions. Recall that the popular CN scheme is also a Runge-Kutta so that its time truncation error is given by\(^{37}\)

\[\epsilon^{(n+c_i)}(\Delta t) = \frac{y^{(n+c_i)} - y^{(n)}}{\Delta t} - \sum_{j=1}^{s} a_{ij} \phi(t^{(n+c_j)}), \text{ for } i = 1, ..., s\]

\(31)

Proposition VIII.1 If the conservative formulation Eq. (5) is used in conjunction with Eqs. (25,26), for spatially linear deformation of triangles or tetrahedra, the Crank-Nicholson scheme will yield the exact solution to the linear shear flow on deforming grids.

See Etienne et al.\(^{34}\) for proof and details.
To reinforce the above conclusions, we complete this analytical study with a numerical time step refinement study. The solution is advanced in time on the time interval $[0, 4]$ starting with one time step $\Delta t$ that we repeatedly divide by 2. Thus

$$\Delta t_k = 4/2^{k-1}, \quad k \in [1, 7]$$

The errors for the velocity and pressure fields are evaluated at time $t = 4$ using the following error norms

$$||e_u||_E^2 = ||u_{ex} - u_h||_E^2 = \int_\Omega (\tau_{ex} - \tau_h) : (\tau_{ex} - \tau_h) d\Omega$$

for the velocity and

$$||e_p||_{L^2}^2 = ||p_{ex} - p_h||_{L^2}^2 = \int_\Omega (p_{ex} - p_h)^2 d\Omega$$

for the pressure. Subscript $ex$ refers to the exact solution and $h$ to the finite element solution.

Fig. 1 and 2 presents results obtained for the Couette Flow using the following mesh motion.

\begin{align*}
x\text{-motion} & \quad \xi(x, y, t) = xyz^2(1 - z)\sin(\pi t/2) \\
y\text{-motion} & \quad \eta(x, y, t) = xy^2z(1 - z)\sin(\pi t/2) \\
z\text{-motion} & \quad \zeta(x, y, t) = x(1 - x)y(1 - y)z(1 - z)\sin(\pi t/2)
\end{align*}

It exhibits a nonlinear (quadratic) dependence in $t$ so that this test will be sensitive to choices made for evaluating $(\nabla \cdot v_d)$ and $v_m$. In particular, $(\nabla \cdot v_d) \neq (\nabla \cdot v_m)$ for IRK3 and IRK5 schemes. Note also that the convergence rates for the other schemes are optimal; that is they are equal to those observed on fixed meshes. These results confirm our analytical error analysis.

Our Crank-Nicholson yields the exact solution because the mesh velocity and the divergence of the mesh velocity are evaluated with Eq. (25, 26). This unusual behavior results from a special property of our
approach. For the ODE $y' = f(t, y)$ it is written as

$$\frac{y^{(n+1)} - y^{(n)}}{\Delta t} = \frac{f(t^{(n)}, y^{(n)}) + f(t^{(n+1)}, y^{(n+1)})}{2}$$

(40)

Because this Crank-Nicholson scheme does not sample the solution at intermediate time steps, it always perceives the mesh motion as a linear function of time; which explains why our CN conservative formulation delivers the exact solution as predicted by proposition VIII.1.

To verify our statement that multi-step methods yield the exact solution to the Couette flow when the mesh motion is a linear function of time, we ran the three schemes with the conservative formulation. All integrators reproduced the exact solution to the Couette flow with velocity and pressure errors of order of machine zero ($\approx 10^{-14}$).

<table>
<thead>
<tr>
<th>Nonlinear $\mathbf{v}_m$</th>
<th>$L_2$-pressure</th>
<th>$H^1$-energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRK1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>SDIRK21</td>
<td>1</td>
<td>1.9</td>
</tr>
<tr>
<td>SDIRK31</td>
<td>1</td>
<td>2.2</td>
</tr>
<tr>
<td>Crank-Nicholson</td>
<td>exact</td>
<td>exact</td>
</tr>
<tr>
<td>IRK3</td>
<td>2</td>
<td>2.7</td>
</tr>
<tr>
<td>IRK5</td>
<td>3</td>
<td>3.9</td>
</tr>
</tbody>
</table>

Table 3. Convergence rate for various time-stepping schemes for the Couette flow and linear/nonlinear time mesh motions.

Table 3 summarizes results for the Couette flow, the different formulations and time integrators with the proposed combination of mesh velocity and mesh velocity divergence. The conservative formulation yields solutions with good time accuracy. Unfortunately, error decreases to machine zero before the reaching the asymptotic range of the scheme and, hence before being able to observe the optimal rates of convergence.
However, at each time-step refinement, rates of convergence improve. We may infer that computations done with more than 14 decimal digits will exhibit optimal rates of convergence.

We note, that the Couette flow does not thoroughly test the viscous terms in the Navier-Stokes equations as the second derivatives of velocity and the convective terms vanish for this flow.

C. The Poiseuille flow

The Poiseuille flow (see Schlichting\cite{30}) provides a test for some of the viscous terms in the Navier-Stokes equations. The velocity component $u$ is a quadratic polynomial in $y$ only, $v$ is zero, and $p$ is linear in $x$ only. The solution is again steady state and independent of $t$ as for the Couette and uniform flows. The problem is made unsteady by using a time varying mesh. We use the same mesh motion as for the Couette flow Eq. (36,39).

The time integration is performed on the time interval $[0,1]$ with the following sequence of time steps

$$\Delta t_k = 1/2^{k-1}, \quad k \in [1,7] \quad (41)$$

Geometry and boundary conditions are depicted on Fig. 3. Again only vertices inside the domain are allowed to move.

![Figure 3. Geometry and boundary conditions for the Poiseuille flow.](image)

Figs. 4 and 5 and Table 4 show that the conservative formulation delivers good rates of convergence for all 6 time integrators. Here, rates of convergence are better than for the Couette flow simply because the error for the coarsest time step is larger for the Poiseuille flow than for the Couette flow. This implies that we need more time-step refinements before reaching machine zero levels of error. There is more room to reach the asymptotic rate of convergence.

<table>
<thead>
<tr>
<th>conservative formulation (5)</th>
<th>Poiseuille $L_2$-pressure</th>
<th>$H^1$-energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>IRK1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>SDIRK21</td>
<td>1</td>
<td>1.95</td>
</tr>
<tr>
<td>SDIRK31</td>
<td>1</td>
<td>2.45</td>
</tr>
<tr>
<td>Crank-Nicholson</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>IKRK3</td>
<td>2</td>
<td>2.11</td>
</tr>
<tr>
<td>IRK5</td>
<td>3</td>
<td>4.03</td>
</tr>
</tbody>
</table>

Table 4. Convergence rate for various time-stepping schemes for the Poiseuille flow and arbitrary mesh motions.

**Remark:** We performed an analytical study similar to that done for the linear shear flow and the CN scheme. For the conservative formulation and Eqs. (26,25), it shows that the time truncation error term is

$$\int_{\Omega(n)} w(-v_m^2/2) \Delta t (J^{(n+1)} - J^{(n)})/J^{(n)} d\Omega$$

which is of order of $O(\Delta t^2)$ and confirms the numerical results. Moreover, this analysis shows that the results are independent of space dimension for linear deforming triangles or tetrahedra.
We note that the Poiseuille flow is not a truly conclusive test for an ALE formulation because the solution is one-dimensional, the convective terms are zero, and the solution is represented exactly by the finite element space discretization. A test case with a more complex flow pattern given by the Method of Manufactured Solutions is required.

D. Manufactured solution

The Method of Manufactured Solutions provides a more exhaustive and general test of the two formulations. In fact the Method of Manufactured Solutions provides for a complete verification of the time-integrators-GCL formulations. Manufactured solutions are constructed in such a way that they exercise all derivatives in the flow equations. The manufactured velocity and pressure solutions are set to

\[
\begin{align*}
  u(x, y, t) &= (1 + x + y + z + x^2 - 2xy + y^2 + yz + z^2 - zx) g(t) \\
  v(x, y, t) &= (1 + x - y/2 + z + x^2 - 2xy + y^2 - yz + z^2 + zx) g(t) \\
  w(x, y, t) &= (1 + x + y - z/2 + x^2 + xy + y^2 + z^2) g(t) \\
  p(x, y, t) &= x^2 \\
  g(t) &= 1 + \tanh(t)
\end{align*}
\]

This solution is represented exactly by the spatial discretization of the P2-P1 Taylor-Hood element used here so that the spatial discretization does not contribute to the overall error. The initial conditions are given by evaluating the manufactured solution at \( t = 0 \). Simulations are performed over the time interval \([0, 0.5]\) with

\[
\Delta t_k = 1/2^k, \quad k \in [1, 12]
\]

Fig. 6 and 7 show temporal convergence rates for the manufactured solution in terms of \( H^1 \) norm of the velocity and \( L^2 \) norm of pressure. The conservative formulation delivers optimal rates of convergence for velocity and pressure as the time step is reduced. Note we did not get the optimal rate for the 5th order IRK. We reach mostly order 4.2. But, as the time step is refined, this order increased. We believe that with higher representation of reals, it will be possible to reach the optimal rate of convergence. Note that this
phenomenon is not unique to simulations on moving meshes. Using the same manufactured case on fixed grids yields very similar curves of error and rates of convergence. This leads to the conclusion that we obtain the same the same convergence rate on deforming and fixed grids (GCL compliance at level 3).

<table>
<thead>
<tr>
<th>conservative formulation (5)</th>
<th>MMS</th>
<th>$L_2$-pressure</th>
<th>$H^1$-energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crank-Nicholson</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>IRK5</td>
<td>3</td>
<td>4.2</td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Convergence rate for various time-stepping schemes for the Manufactured flow and arbitrary mesh motions.

IX. Conclusions

This paper presented a new approach for constructing GCL compliant time accurate ALE Finite Element Methods for viscous incompressible flows. The approach consists in decoupling the construction of the mesh velocity divergence ($\nabla \cdot \mathbf{v}_d$) from the construction of the mesh velocity itself $\mathbf{v}_m$. This provides the flexibility to achieve GCL compliance and to maintain the temporal accuracy of the fixed mesh time-integrator when applied to flows on deforming domains. The approach is applicable to a broad spectrum of time integrators ranging from the commonly used first order implicit backward Euler scheme through the very popular second-order Crank-Nicholson scheme to the more sophisticated 2nd and 3rd Singly Diagonally Implicit Rung-Kutta and the 3rd and 5th order Implicit Runge-Kutta schemes.

Analytical solutions and time step refinement studies were used to determine under what conditions the conservative weak form of the Navier-Stokes equations satisfies the GCL and maintains the fixed mesh high order temporal accuracy of the time integrator when applied to flows on deforming domains.

For the conservative formulation, construction of the mesh velocity divergence determines GCL compliance while the method for evaluating the mesh velocity is responsible for maintaining the high order temporal accuracy of the fixed mesh time integrator when used on deforming meshes (GCL, level 3).

The formulas developed in this work are straightforward to implement, result in simpler codes that are
Figure 6. Grid convergence for the manufactured solution, $H^1$ energy norm.

easier to modify and maintain because the same time-integrator can be used for two- or three-dimensional flows and for fixed and deforming domains. Optimal temporal accuracy on deforming domains was demonstrated analytically on simple flows and confirmed by thorough time-step refinement studies. Note that in many cases double precision representation of reals does not allow us to get optimal rates of convergence. In these cases we observed the same behavior of convergence rates on fixed grids.

The present numerical results confirm that our two mesh-velocity approach to finite element ALE computation proven and tested in 2-D is also valid for 3-D problems.

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References

Figure 7. Grid convergence for the manufactured solution, $L_2$ pressure norm.


