Generalized finite compact difference scheme for shock/complex flowfield interaction

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Abstract

A class of generalized high order finite compact difference schemes is proposed for shock/vortex, shock/boundary layer interaction problems. The finite compact difference scheme takes the region between two shocks as a compact stencil. The high order WENO fluxes on shock stencils are used as the internal boundary fluxes for the compact scheme. A lemma of the property of smoothness estimators on a 5-points stencil is given to detect the shock position. There is no free parameter introduced. Some numerical experiments are given as the application of the new scheme.

1 Introduction

The numerical schemes for direct numerical simulation (DNS) and large eddy simulation (LES) of interaction of shock waves/complex flows need to resolve both shock waves and fine flow structures. Since weighted essentially non-oscillatory (WENO) schemes tend to have uniform higher order accuracy in smooth region and keep the essentially non-oscillatory properties near shock waves, they are widely used in simulation of flows with discontinuities such as shock waves and contact surfaces[1]. However, even though the order of accuracy for WENO schemes can be designed to be arbitrarily high[2], such as the eleven order WENO scheme develop by Balsara and Shu[3], the resolution of short waves of WENO schemes in smooth regions tend to be more diffusive than central differencing schemes.

The best method to simulate wave dominated problems is the spectral method[4], but it is limited to simple geometries with generally periodic boundary conditions. Due to compact schemes’ spectral-like resolution, and their flexibility, compact schemes[5] have attracted more and more attention and have been widely used to DNS and LES of turbulence flow[6, 7, 8, 9, 10, 11]. A drawback of compact schemes is that they will generate Gibbs-like oscillations around shock waves or large gradients.

In recent years, there are many efforts to make compact schemes obtain shock-capturing capability. Cockburn and Shu[12] developed the nonlinear stable compact schemes using the TVDM(total variation diminishing in the means) property. The schemes require an implicit symmetric matrix and a reconstruction from the mean variable obtained by TVDM compact schemes. Ravichandran[13] improved this type of schemes and a class of the compact upwind schemes was developed without the limitation of a symmetric matrix. Tu and Yuan[14] constructed a fifth-order shock-capturing compact upwind scheme by using a characteristic-based flux splitting limited method.

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Adams and Shariff[15] proposed a hybrid compact-ENO scheme for shock-turbulence interaction problems. Following the same basic approach, Pirozzoli[16] derived a conservative hybrid compact-WENO scheme. Ren et al.[17] presented a fifth-order conservative hybrid compact-WENO scheme for shock-capturing calculation, which is constructed through the weighted average of conservative compact scheme and WENO scheme. Zhou et al. [18] suggested a new family of high order compact upwind difference schemes, which are later made to have shock-capturing capability by hybridizing them with WENO schemes.

Shen et al.[19] proposed a finite compact scheme, which treats the discontinuity as the internal boundary point, hence, avoids the global dependence of the traditional compact schemes. Combined with the TVD of ENO limiters, a set of high resolution finite compact (FC) difference schemes with only bi-diagonal matrix inversion are constructed[19, 20].

Some of the above methods need to calculate the preliminary fluxes first by using a compact scheme[12, 13, 14, 15, 16]. This will result in contaminated (oscillatory) fluxes in the regions near discontinuities, and hence the compact fluxes will lose their high order accuracy in those regions. The other methods that use the limiter functions will degrade the accuracy at extrema to first order[12, 13, 14, 19]. All the hybrid schemes[12, 15, 16, 17, 18, 19] introduce a free parameter to switch to the ENO/WENO schemes at discontinuities. The weighted compact nonlinear schemes proposed by Deng and Zhang[21] need to use more nodes than a standard compact scheme and hence lose the compactness. The same problem exists in the higher order extensions[2] of Deng and Zhang’s method.

Artificial dissipation and compact filters are also introduced into compact scheme to help stabilize numerical solutions and reduce oscillations near discontinuities[11, 22, 23, 24, 25, 26, 27]. Nonlinear characteristics-based (artificial compression method, ACM) filters is used to construct the low-dissipative high order shock-capturing scheme by Yee et al.[28] with a problem dependent parameter introduced. A WENO-type smoothness estimator is used in [29] as a sensor to switch between the high-order compact spatial filters and the ACM filters, for which a threshold parameter for the sensor is also needed.

For the WENO schemes first proposed by Liu et al.[30] and improved by Jiang and Shu[31], a small difference between the smoothness estimators can reduce the numerical accuracy. A function to decrease the weights sensitivity to the smoothness estimators variation is used by Shen and Zha[32] to improve the WENO scheme accuracy. In addition, when it is applied to practical flows, the original value of the parameter $\varepsilon$ used to construct the weight functions will generate oscillatory weights, which result in convergence and accuracy problems. An optimized $\varepsilon$ value of $10^{-2}$ is suggested by Shen et al.[33] to improve convergence and accuracy. On the other hand, Henrick et al.[34] pointed out that the original smoothness indicators of Jiang and Shu fail to improving the accuracy order of WENO scheme at critical points, where the first derivatives are zero. A mapping function is proposed by Henrick et al.[34] to obtain the optimal order near critical points. Recently, Borges et al.[35] suggest to use the whole 5-points stencil to devise a smoothness indicator of higher order than the classical smoothness indicator proposed by Jiang and Shu[31]. Shen and Zha[36] used this smoothness estimator to improve the accuracy at transition points of a discontinuity.

Based on the results of Borges et al.[35], this paper gives a lemma about the property of smoothness estimators on whole the 5-points stencil. A class of generalized finite compact difference schemes is proposed by using the lemma to detect the shock position. There is no free parameter introduced.

## 2 The Numerical Algorithm

For the hyperbolic conservation law in the form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad (1)$$
the semi-discretization form can be written as
\[
\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}})
\]  

(2)

2.1 Weighted essentially non-oscillatory (WENO) Scheme

The flux \(h_{i+\frac{1}{2}}\) of the classical fifth-order WENO scheme \([31, 35]\) is built through the convex combination of interpolated values \(\hat{f}^k(x_{i+\frac{1}{2}})(k = 0, 1, 2)\), in which \(\hat{f}^k(x)\) is the third degree interpolation polynomial on stencil \(S^3_k = (x_{i+k-2}, x_{i+k-1}, x_{i+k})\),

\[
h_{i+\frac{1}{2}} = \sum_{k=0}^{2} \omega_k \hat{f}^k(x_{i+\frac{1}{2}})
\]

where

\[
\hat{f}^k(x_{i+\frac{1}{2}}) = \hat{f}_{i+\frac{1}{2}} = \sum_{j=0}^{2} c_{kj} f_{i-k+j}, \quad i = 0, \ldots, N
\]

The weights \(\omega_k\) are defined as

\[
\omega_k = \frac{\alpha_k}{\sum_{l=0}^{3} \alpha_l}, \quad \alpha_k = \frac{d_k}{(\beta_k + \varepsilon)^p}
\]

(5)

The smoothness indicators \(\beta_k\) are given by[31]

\[
\beta_k = \sum_{l=1}^{2} \Delta x^{2l-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \frac{d^l}{dx^l} \hat{f}^k(x) \right)^2 dx
\]

(6)

Taylor expansion of (6) gives

\[
\begin{align*}
\beta_0 &= \frac{13}{12} (f_{i-2} - 2f_{i-1} + f_i)^2 + \frac{1}{4} (f_{i-2} - 4f_{i-1} + 3f_i)^2 \\
\beta_1 &= \frac{13}{12} (f_{i-1} - 2f_i + f_{i+1})^2 + \frac{1}{4} (f_{i-1} - f_{i+1})^2 \\
\beta_2 &= \frac{13}{12} (f_i - 2f_{i+1} + f_{i+2})^2 + \frac{1}{4} (3f_i - 4f_{i+1} + f_{i+2})^2
\end{align*}
\]

(7)

Henrick et al[34] shown that if \(\beta_k\) satisfy \(\beta_k = D(1 + O(\Delta x^3))\), the weights \(\omega_k\) then satisfy \(\omega_k = d_k + O(\Delta x^3)\), where \(D\) is some non-zero quantity independent of \(k\). And the necessary and sufficient conditions for fifth-order convergence in (2) are given as[34]:

\[
\sum_{k=0}^{2} A_k(\omega^+ - \omega^-) = O(\Delta x^3)
\]

(8)

\[
\omega_k^+ - d_k = O(\Delta x^2)
\]

(9)

A sufficient condition for fifth-order of convergence is given by[35]:

\[
\omega_k^+ - d_k = O(\Delta x^3)
\]

(10)

If \(f_i' = 0\), Eq.(7) gives \(\beta_k = D(1 + O(\Delta x))\) and \(\omega_k = d_k + O(\Delta x)\), this will degrade the convergence accuracy of the scheme. Shen et al [37] proposed a step-by-step reconstruction method, in which two 4th order weighted fluxes obtained from 3rd ENO fluxes are used to construct 5th order WENO scheme. Henrick et al[34] proposed a mapping function to increase the approximation of \(\omega_k\) to the ideal weights \(d_k\).
weights that satisfies the necessary and sufficient conditions for fifth-order convergence. The new smoothness indicators $\beta_k^z$ defined by Borges et al [35] are

$$\beta_k^z = \frac{\beta_k + \varepsilon}{\beta_k + \tau_5 + \varepsilon}, \quad k = 0, 1, 2$$

and the new WENO weights $\omega_k^z$ are

$$\omega_k^z = \frac{\alpha_k^z}{\sum_{l=0}^{d_k} \alpha_l^z}, \quad \alpha_k^z = \frac{d_k}{\beta_k^z} = d_k \left( 1 + \left( \frac{\tau_5}{\beta_k + \varepsilon} \right)^q \right), \quad k = 0, 1, 2$$

where

$$\tau_5 = |\beta_0 - \beta_2|$$

The coefficients $c_{kj}$ and $d_k$ are listed in Table 1. The parameter $\varepsilon$ is used to avoid the division by zero ($\varepsilon = 10^{-6}$ is used in [31] and $\varepsilon = 10^{-40}$ is used in [35]), $p$ in (5) and $q$ in (12) are chosen to increase the difference of scales of distinct weights at non-smooth parts of the solution. As pointed out by Borges et al[35], for a smooth function, increasing the value of $q$ in Eq. (12) decreases the correction of the WENO-Z weights to the ideal weights $d_k$, making the scheme closer to the optimal central scheme; on the other hand, increasing $q$ also decreases the relative importance of the discontinuous substencil and makes the scheme more dissipative.

### 2.2 The important properties of $\tau_5$

Taylor series expansions of $\beta_k$, Eq.(7), at $x_i$ are

$$\beta_0 = f_i^2 \Delta x^2 + \left( \frac{13}{12} f_i^2 - \frac{2}{3} f_i f_i^\prime \right) \Delta x^4 - \left( \frac{13}{6} f_i^4 f_i^\prime + \frac{1}{2} f_i f_i^{(4)} \right) \Delta x^5 + \frac{43}{36} f_i^6 + \frac{91}{72} f_i^5 f_i^\prime + \frac{7}{30} f_i^4 f_i^{(5)} \Delta x^6 + O(\Delta x^7)$$

$$\beta_1 = f_i^2 \Delta x^2 + \left( \frac{13}{12} f_i^2 - \frac{3}{2} f_i f_i^\prime \right) \Delta x^4 + \left( \frac{13}{6} f_i^4 f_i^\prime + \frac{7}{12} f_i^5 f_i^\prime + \frac{1}{2} f_i^4 f_i^{(4)} + \frac{1}{120} f_i^6 f_i^{(5)} \right) \Delta x^6 + O(\Delta x^8)$$

$$\beta_2 = f_i^2 \Delta x^2 + \left( \frac{13}{12} f_i^2 - \frac{3}{2} f_i f_i^\prime \right) \Delta x^4 + \left( \frac{13}{6} f_i^4 f_i^\prime + \frac{7}{12} f_i^5 f_i^\prime - \frac{1}{2} f_i^4 f_i^{(4)} \right) \Delta x^5 + \frac{43}{36} f_i^6 + \frac{91}{72} f_i^5 f_i^\prime - \frac{7}{30} f_i^4 f_i^{(5)} \Delta x^6 + O(\Delta x^7)$$

From Eqs. (13) and (14), it is easy to find that

$$\tau_5 = |\beta_0 - \beta_2| = \frac{13}{3} |f_i^4 f_i^{(4)}| \Delta x^5 + O(\Delta x^7)$$

In fact, it can be proved that $\beta_0$ and $\beta_2$ have the same even order terms. And $\tau_5$ is at least one order higher than $\beta_k (k = 0, 1, 2)$. For example, if $f_i^\prime = f_i^{(4)} = 0$ and $f_i^\prime = f_i^{(4)} \neq 0$, then $\beta_k = A_k \Delta x^6 + O(\Delta x^7)$, $A_0 = A_2$ and $\tau_5 = B \Delta x^7 + O(\Delta x^9)$. And so on. Where, $A_k$ and $B$ are constants independent of $\Delta x$. 

<table>
<thead>
<tr>
<th>$k$</th>
<th>$j=0$</th>
<th>$j=1$</th>
<th>$j=2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1/3$</td>
<td>$-7/6$</td>
<td>$11/6$</td>
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<tr>
<td>1</td>
<td>$-1/6$</td>
<td>$5/6$</td>
<td>$1/3$</td>
</tr>
<tr>
<td>2</td>
<td>$1/3$</td>
<td>$5/6$</td>
<td>$-1/6$</td>
</tr>
</tbody>
</table>

Table 1: Coefficients $c_{kj}$ and $d_k$
For completeness, the important properties of $\tau_5$[35] are listed below:

1. For a stencil $S^5 = (x_{i-2}, x_{i-1}, \cdots, x_{i+2})$, if $S^5$ does not contain discontinuities, then $\tau_5 << \beta_k$ for $k = 0, 1, 2$;

2. if the solution is continuous at some of the stencil $S^3_k$, but discontinuous in the whole stencil $S^5$, then for the smooth stencils, $\beta_k << \tau_5$.

3. $\tau_5 \leq \text{max}(\beta_0, \beta_1, \beta_2)$.

Following the above properties, it is straightforward to prove the following lemma:

**Lemma 1.** If $\tau_5 > \min(\beta_0, \beta_1, \beta_2)$, then $S^5$ is an unsmooth stencil.

### 2.3 The Finite Compact Differencing Scheme

The general form of compact difference scheme[5] for the derivative $f_i' = \frac{\partial f}{\partial x}$ can be written as

$$Af_i' = \frac{1}{\Delta x} B f_i$$

(16)

or use the flux form by using

$$f_i' = \frac{1}{\Delta x} (\hat{h}_{i+1/2} - \hat{h}_{i-1/2})$$

$$A\hat{h}_{i+1/2} = C f_i, \; i = 1, 2, \cdots, N$$

(17)

where $A$, $B$ and $C$ are the coefficient matrices depending on different requirements.

In [19, 20], finite compact difference is obtained by using forward marching (bi-diagonal matrix) and TVD/ENO limiters. First, the bi-diagonal matrix $A$ is used as,

$$\alpha h_{i-1/2} + \beta h_{i+1/2} = \sum_{l=n_1}^{n_2} a_l f_{i+l}$$

(18)

The coefficients $\alpha$, $\beta$ and $a_l$ can be found in [19, 20].

Different from the traditional compact difference scheme (17), in Eq. (18), $h_{i-1/2}$ is used rather than $\hat{h}_{i-1/2}$. According to the construction of $h_{i-1/2}$[19, 20], in general, there is always $h_{i-1/2} = \hat{h}_{i-1/2}$ in smooth region; and near discontinuity, $h_{i-1/2}$ is a flux with essentially non-oscillatory property. That is, if the stencil involving the interface $i - 1/2$ is a discontinuous stencil, then the essentially non-oscillatory flux $h_{i-1/2}$ is used to calculate $\hat{h}_{i+1/2}$. This prevents a "compact" scheme from crossing the discontinuity, and hence avoids the effect of the error or oscillation introduced by $\hat{h}_{i-1/2}$ on $\hat{h}_{i+1/2}$. The global compact difference scheme (17) is divided into many compact difference schemes by discontinuities, hence the scheme (18) is called finite compact difference scheme[19, 20].

In this paper, a generalized finite compact difference is suggested as the following,

1. the discontinuous interface is detected by using Lemma 1, and the flux on this interface is calculated by a WENO scheme.

2. those point(s) between two discontinuous interfaces(or between boundary point and discontinuous interface)are defined as a compact stencil, and the compact scheme is used to solve the numerical fluxes on the compact stencil.

Note: in step (2), the fluxes on the discontinuous interfaces obtained from step (1) are automatically used as the internal boundary fluxes.

Fig. 1 is a sketch showing this finite compact scheme. $M$ denotes the number of compact stencil divided by discontinuities.
2.4 Procedure of the Finite Compact Differencing Scheme

BEGIN

Step 0. calculate $h_{1/2}$ and $h_{N+1/2}$ using boundary formula

Step 1. M=1 ! looking for starting and ending points for compact scheme using Lemma 1

start_point(M)=1 ! compact stencil starting point

DO i=1,N-1

   calculate $\beta_0, \beta_1, \beta_2$

   $\tau_5 = |\beta_0 - \beta_2|$

   IF $\tau_5 > \min(\beta_0, \beta_1, \beta_2)$ THEN

      end_point(M)=i ! compact stencil ending point

      M=M+1

      start_point(M)=i+1

   calculate $h_{i+1/2}$ using WENO scheme:

   $h_{i+1/2} = \sum_{k=0}^{2} \omega_k f^k(x_{i+1/2})$ (19)

END IF

END DO

end_point(M)=N

Step 2. DO k=1,M ! calculate fluxes on compact stencil(smooth region) $k$ using a compact scheme

DO j=start_point(k), end_point(k)-1

   calculate the right hand side $d_{i+1/2}$ of compact scheme

END DO

solve tridiagonal compact scheme:

$\alpha h_{i-1/2} + \gamma h_{i+1/2} + \beta h_{i+3/2} = d_{i+1/2},$ (20)

where, $j =$start_point(k), $\cdots$, end_point(k)-1.

END DO

END

For step 1 and 2, any WENO and compact schemes can be used to construct the fluxes. In this paper, the fifth-order WENO scheme (WENO-Z) suggested by Borges et al.[35] and the 6th-order Pade scheme[5] are suggested for their high accuracy. Hence, in Eq. (20),

$$\alpha = \beta = \frac{1}{3}, \gamma = 1$$

and

$$d_{i+1/2} = \frac{1}{36}(f_{i+2} + 29f_{i+1} + 29f_i + f_{i-1})$$
Table 2: Accuracy on $u_t + u_x = 0$ with $u_0(x) = \sin(2\pi x)$, $t=1$

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$N$</th>
<th>$L_{\infty}$ error</th>
<th>$L_{\infty}$ order</th>
<th>$L_1$ error</th>
<th>$L_1$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>WENO-Z</td>
<td>20</td>
<td>0.110652E-01</td>
<td>5.015</td>
<td>0.754065E-02</td>
<td>5.165</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.342283E-03</td>
<td>5.067</td>
<td>0.210204E-03</td>
<td>5.059</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.102111E-04</td>
<td>5.022</td>
<td>0.197966E-06</td>
<td>4.994</td>
</tr>
<tr>
<td></td>
<td>160</td>
<td>0.314222E-06</td>
<td>5.004</td>
<td>0.620628E-08</td>
<td>4.995</td>
</tr>
<tr>
<td></td>
<td>320</td>
<td>0.979162E-08</td>
<td>5.004</td>
<td>0.181680E-09</td>
<td>5.032</td>
</tr>
<tr>
<td>present</td>
<td>20</td>
<td>0.578082E-02</td>
<td>5.015</td>
<td>0.152544E-02</td>
<td>4.992</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.133000E-03</td>
<td>5.442</td>
<td>0.405741E-04</td>
<td>5.233</td>
</tr>
<tr>
<td></td>
<td>80</td>
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<td>5.941</td>
<td>0.181680E-09</td>
<td>6.032</td>
</tr>
</tbody>
</table>

Different formula can be used to calculate the boundary fluxes $h_{1/2}$ and $h_{N+1/2}$. For example, the biased sixth order formula are used in this paper to calculate the boundary fluxes $h_{1/2}$ and $h_{N+1/2}$,

$$h_{1/2} = \frac{1}{60}(-2f_{-1} + 22f_0 + 57f_1 - 23f_3 + 7f_4 - f_5)$$

$$h_{N+1/2} = \frac{1}{60}(-f_{N-3} + 7f_{N-2} - 23f_{N-1} + 57f_N + 22f_{N+1} - 2f_{N+2})$$

The finite compact scheme(18)[19, 20] can only use the compact scheme with bi-diagonal matrix $A$(upwind-type), whereas the present generalized finite compact scheme can used any compact scheme, for example, the Pade scheme (20). Thus, using the same points, the present scheme can achieve higher order accuracy.

### 3 Numerical Examples

In this paper, the 4th order Runge-Kutta-type method[38] is used for the time marching.

#### 3.1 Linear transport equation

The linear transport equation is used to test the accuracy of WENO schemes.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad -1 < x < 1$$

$$u(x, 0) = u_0(x), \text{ periodic}$$

(1)

$$u_0(x) = \sin(2\pi x)$$

This smooth solution is used to validate the scheme accuracy. Table 2 given the error and accuracy order. It can be seen that, for smooth solution, the present scheme obtains sixth order accuracy, which is one order more than WENO-Z scheme.

(2)

$$u_0(x) = \begin{cases} 
-x\sin(3\pi x^2/2), & -1 \leq x < -1/3 \\
|\sin(2\pi x)|, & -1/3 \leq x \leq 1/3 \\
2x - 1 - \sin(3\pi x)/6, & \text{otherwise}
\end{cases}$$
Fig. 2 shows the numerical solutions at \( t = 6 \). It can be seen that the present method resolve the peaks and discontinuities slopes more accurately than the WENO-Z scheme of Borges et al [35].

\[
\begin{cases}
\frac{1}{6}(G(x, \beta, z - \delta) + G(x, \beta, z + \delta) + 4G(x, \beta, z)), & -0.8 \leq x \leq -0.6, \\
1, & -0.4 \leq x \leq -0.2, \\
1 - |10(x - 0.1)|, & 0 \leq x \leq 0.2, \\
\frac{1}{6}(F(x, \alpha, \alpha - \delta) + F(x, \alpha, \alpha + \delta) + 4F(x, \alpha, \alpha)), & 0.4 \leq x \leq 0.6, \\
0, & \text{otherwise}
\end{cases}
\]

Same as in Ref. [31], the constants for this case are taken as \( a = 0.5, z = -0.7, \delta = 0.005, \alpha = 10, \) and \( \beta = \log 2/36\delta^2 \). The solution contains a smooth combination of Gaussians, a square wave, a sharp triangle wave, and a half ellipse.

The results at \( t = 6 \) with 200 grid points are shown in Figs. 3 and 4. It can be seen that, the present method can improve the accuracy not only for the continuous solution (Gaussians, sharp triangle wave, and half ellipse), but also for the discontinuous waves (square wave).

### 3.2 1D Shock Wave Tube, Sod Problem

To examine the new scheme for nonlinear equations, the one-dimensional Euler equations are solved for the 1D shock tube problem.

1D Euler equations:

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0
\]

where

\[
U = \begin{bmatrix}
p \\ \rho u \\ \rho e
\end{bmatrix}, \\
F = \begin{bmatrix}
\rho u \\ \rho u^2 + p \\ u(\rho e + p)
\end{bmatrix}, \quad p = (\gamma - 1)(\rho e - \rho u^2/2), \quad \gamma = 1.4.
\]

The initial condition is

\[
(\rho, u, p) = \begin{cases}
(1.0, 0.0, 1.0), & x \leq 7.5, \\
(0.125, 0.0, 0.1), & x > 7.5.
\end{cases}
\]

In this paper, the Steger-Warming flux vector splitting method [39] is used. The grid points is \( N = 200 \). Fig. 5 gives the density and velocity distribution. Both the WENO-Z and present schemes capture the shock very well, even though both schemes have a small oscillation at the tail of the contact discontinuity.

### 3.3 1D Shock Wave Tube, Shu-Osher Problem

This problem is governed by the one-dimensional Euler equations (22) with following initial condition:

\[
(\rho, u, p) = \begin{cases}
(3.857143, 2.629369, 10.3333), & when \ x < -4, \\
(1 + \varepsilon \sin(5x), 0.0, 1.0), & when \ x \geq -4.
\end{cases}
\]

where, \( \varepsilon = 0.2 \). This case represents a Mach 3 shock wave interacting with a sine entropy wave [40]. The results at time \( t = 1.8 \) with mesh size of 200 are plotted in Figs. 6 and 7. The “exact” solutions are the numerical solutions of the original WENO-5 scheme with grid points of \( N = 8000 \). For this case, it can be seen that the present scheme resolves the wave peaks significantly better than the WENO-Z scheme due to the smaller dissipation of compact scheme.
3.4 Two-dimensional Shock Vortex Interaction

A two-dimensional shock vortex interaction problem is solved to demonstrate the scheme for multidimensional flows. The two-dimensional Euler equations are solved for this problem:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0$$  (25)

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho e + p) \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho e + p) \end{bmatrix},$$

$$p = (\gamma - 1)(\rho e - \rho(u^2 + v^2)/2), \quad \gamma = 1.4.$$  

The problem is taken from Ref. [31]. It describes the interaction between a stationary shock and a vortex. The computational domain is taken to be $[0; 2] \times [0; 1]$. A stationary Mach 1.1 shock is positioned at $x = 0.5$ and normal to the x-axis. Its left state is $(\rho, u, v, p) = (1, 1.1\sqrt{\gamma}, 0, 1)$. A small vortex is superimposed to the flow on the left of the shock and is centered at $(x_c, y_c) = (0.25, 0.5)$. The vortex is described as a perturbation to the velocity $(u, v)$, temperature $(T = p/\rho)$, and entropy $(S = \ln(p/\rho\gamma))$ of the mean flow and denoted by the tilde values:

$$\tilde{u} = \varepsilon \tau e^{a(1 - \tau^2)} \sin\theta$$

$$\tilde{v} = -\varepsilon \tau e^{a(1 - \tau^2)} \cos\theta$$

$$\tilde{T} = -\frac{(\gamma - 1)\varepsilon^2 e^{2a(1 - \gamma^2)}}{4a\gamma}$$

$$\tilde{S} = 0$$

where $\tau = r/r_c$ and $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$, $\varepsilon$ indicates the strength of the vortex, $a$ controls the decay rate of the vortex, and $r_c$ is the critical radius for which the vortex has the maximum strength. As in the Refs. [31, 20], $\varepsilon = 0.3$, $r_c = 0.05$, and $a = 0.204$ are adopted in this paper.

The time step is taken as follows [16]:

$$\Delta t = \delta \frac{\Delta t_x \Delta t_y}{\Delta t_x + \Delta t_y}, \text{ with } \Delta t_x = \frac{\Delta x}{\max_{i,j}(|u_{i,j}| + c_{i,j})}, \Delta t_y = \frac{\Delta y}{\max_{i,j}(|v_{i,j}| + c_{i,j})}$$  (26)

where $\delta = 0.5$ is the CFL number.

Fig. 8 is the pressure contours at $t = 0.60$. Figs. 9 and 10 are the comparisons of the pressure between the present and the WENO-Z scheme [35] along the center line at $y = 0.5$. In order to show the accuracy of the new scheme, the results obtained by the WENO-Z scheme with a refined mesh of $1001 \times 401$ is given as the “exact” solution. With the same coarse mesh density of $251 \times 101$, the new scheme obtains more accurate results than the WENO-Z scheme. Fig. 9 also shows that the new scheme has the sharper shock profile. Fig. 10 indicates that the new scheme achieves lower vortex core pressure due to lower numerical dissipation.

4 Conclusions

An important lemma of the property of smoothness estimators on a 5-points stencil is given and used to formulate a class of generalized finite compact difference schemes. Based on the relationship between the smoothness estimators of the WENO scheme [31] and the parameter $\tau_5$, which is defined as the absolute
difference between the smoothness estimators $\beta_0$ and $\beta_2$ by Borges et al. [35], the lemma is used to detect a discontinuous stencil. A compact central differencing scheme is then used in the smooth region between two discontinuities, which are resolved by a WENO scheme. There is no free parameter introduced. The lemma can also be applied to other methods which need to detect discontinuity locations.

In this paper, the fifth-order WENO-Z scheme and sixth-order Pade scheme are used to construct the finite compact difference scheme. Hence, the present scheme is sixth-order accurate in smooth region and has the essentially non-oscillatory property as WENO scheme near discontinuities. The numerical examples show that the new finite compact scheme is less diffusive than the WENO-Z scheme due to its central differencing nature in smooth regions.

5 Acknowledgment

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References


Figure 1: The sketch of the finite compact scheme

Figure 2: Numerical results, t=6

Figure 3: Numerical results, t=6

Figure 4: Locally enlarged plot of Fig. 3
Figure 5: Density distribution, Sod problem

Figure 6: Density distribution, Shu-Osher problem

Figure 7: Zoomed plot of Fig. 6, Shu-Osher problem

Figure 8: The pressure contours of present scheme, $t = 0.60$
Figure 9: Comparison of pressure at the central line downstream the shock, $t = 0.60$

Figure 10: Comparison of pressure at the central line cross the vortex, $t = 0.60$