Adaptive Disturbance Tracking Control for Large Horizontal Axis Wind Turbines with Disturbance Estimator in Region II Operation

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A new control problem called Disturbance Tracking Control (DTC), which arises in active control of variable speed horizontal axis wind turbines for electric power generation, was developed previously. Feedback control of a linear plant, which is persistently disturbed, must cause the plant output to track a linear function of the disturbance. This control theory is related to Tip Speed Ratio Tracking for wind turbines operating in Region II. The DTC approach was developed for fixed gain controllers where the parameters of the turbine are very well known.

An adaptive version of the DTC Theory for turbines with poorly known parameters was developed previously. However, the adaptive DTC needs measurement of actual wind speed. In this paper we augmented a wind speed estimator and apply the theory to create an adaptive tip speed ratio tracking controller for a horizontal axis wind turbine generator based upon a model of the NREL Controls Advanced Research Turbine (CART).

Nomenclature

\[
\begin{align*}
R & = \text{radius of the turbine blades} \\
\dot{\lambda} & = \text{Tip-speed ratio} \\
\lambda_{op} & = \text{Desired operating point of Tip-speed ratio} \\
\omega & = \text{Hub-height wind speed} \\
\omega_r & = \text{Turbine rotor angular speed} \\
\omega_{op} & = \text{Desired operating point of Hub-height wind speed} \\
\omega_r^{op} & = \text{Desired operating point of Turbine rotor angular speed} \\
U & = \text{Linear plant control inputs} \\
\mathbf{u}_D & = \text{Linear plant disturbance input(s)} \\
\mathbf{u}_D & = \text{Estimated plant disturbance input(s)} \\
\mathbf{x}_p & = \text{Linear plant states} \\
\mathbf{y}_p & = \text{Linear plant outputs} \\
\mathbf{y}_e & = \text{Output Tracking Error} \\
\mathbf{G}_e, \mathbf{G}_D & = \text{Adaptive gain matrices of the appropriate compatible dimensions} \\
\gamma_e, \gamma_D & = \text{Arbitrary, positive definite matrices}
\end{align*}
\]

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I. Introduction

Large wind turbines are operated in three regions. Region I is the turbine start up; Region III is the turbine operated at rated power, and Region II is the turbine operating in between Regions I and III with enough wind to generate power but not at full rated capacity. In Region III blade pitch control is used to achieve constant rotor speed, and in region II generator torque is controlled via the power electronics to produce a constant tip speed ratio. This paper will focus entirely on Region II operation where a constant tip speed ratio produces the best results.

A theory of Adaptive Disturbance Tracking Control (DTC) was developed by Balas in [1] to track the actual wind speed in Region II control problem for Horizontal Axis Wind Turbine (HAWT) electric power generation. In this paper we developed a wind speed estimator and used in Adaptive DTC. Simulation results will be presented on a dynamic model of the NREL Controls Advanced Research Turbine (CART).

An approach to the reduction or counteraction of persistent disturbances was developed by Johnson in [2]. In [3], Balas developed the idea of disturbance tracking control, and Stol modified this in [4]. The motivation for DTC comes from the desire to use control to track some of the wind disturbance. Performance of the wind turbine in Region II is dependent upon the Tip Speed Ratio (TSR):

\[ \lambda \equiv \frac{\omega_T R}{\omega} \]

where \( \omega \) is the wind speed flowing into the turbine blades, \( \omega_T \) is the turbine rotor angular speed, \( R \) is the radius of the turbine blades. Best performance in Region II is obtained by keeping this TSR constant.

We actually linearize the TSR Error:

\[ \varepsilon \equiv \lambda - \lambda_{op} = \frac{\omega_T R}{\omega} - \frac{\omega_{T_{op}} R}{\omega_{op}} \approx \frac{R}{\omega_{op}} [\Delta \omega_T - \omega_{T_{op}}] - \Delta \omega \] (1)

where \( \Delta \omega_T = \omega_T - \omega_{T_{op}}, \Delta \omega = \omega - \omega_{op}, \) and \((\omega_{T_{op}}, \omega_{op})\) is the desired turbine operating point corresponding to the desired TSR \( \lambda_{op} \). We let the Output Tracking Error be

\[ \left\{ \begin{align*}
    e_y &\equiv \Delta \omega_T - Q \Delta \omega \\
    \text{with } Q &\equiv \frac{\omega_{T_{op}}}{\omega_{op}}
\end{align*} \right. \] (2)

and think of \( \Delta \omega_T \), the turbine speed variation, as a measured output of the turbine and \( \Delta \omega \), the wind speed fluctuations, as a disturbance on the turbine. Then DTC becomes choosing a feedback control law that produces:

\[ e_y \equiv \Delta \omega_T - Q \Delta \omega \xrightarrow{t \to \infty} 0 \] (3)

This approximately produces tracking of the desired TSR: \( \varepsilon \equiv \lambda - \lambda_{op} \xrightarrow{t \to \infty} 0 \).

II. Adaptive Disturbance Tracking Control Theory with Persistent Disturbances

The design of the Region II adaptive torque controller makes use of a direct adaptive control approach with adaptive tracking of persistent disturbances. In this section we develop the general adaptive DTC theory. In [5], we presented an adaptive approach to turbine speed regulation for Region III using similar ideas, but the overall results are different for adaptive DTC.

The plant is assumed to be well modeled by the linear, time-invariant, finite-dimensional system:

\[ \begin{cases}
    \dot{x}_p = Ax_p + Bu_p + \Gamma u_D \\
    y_p = Cx_p; \quad x_p(0) = x_0
\end{cases} \] (4)
where the plant state, \( x_p(t) \), is an \( N_p \)-dimensional vector, the control input vector, \( u_p(t) \), is \( M \)-dimensional, and the sensor output vector, \( y_p(t) \), is \( P \)-dimensional. The disturbance input vector, \( u_D(t) \), is \( M_D \)-dimensional and will be thought to come from the Disturbance Generator:

\[
\begin{align*}
\dot{u}_o &= \Theta z_D \\
\dot{z} &= Fz_D; z_D(0) = z_0
\end{align*}
\]  

(5)

where the disturbance state, \( z_D(t) \), is \( N_D \)-dimensional. All matrices in Eqs. (4)-(5) have the appropriate compatible dimensions. Such descriptions of persistent disturbances were first used in [1] to describe signals of known form but unknown amplitude. Equation (5) can be rewritten in a form that is not a dynamical system, which is sometimes easier to use:

\[
\begin{align*}
u_D &= \Theta z_D \\
z_D &= L\phi_D
\end{align*}
\]  

(6)

where \( \phi_D \) is a vector composed of the known basis functions for the solution of \( u_D = \Theta z_D \), i.e., \( \phi_D \) are the basis functions which make up the known form of the disturbance, and \( L \) is a matrix of appropriate dimension.

If the parameters \( \Theta \) and \( F \) of the disturbance are known in equation (5) then disturbance can be estimated from the output using following estimator:

\[
\begin{align*}
\dot{z}_D &= Fz_D + K_D \hat{e}_y \\
\hat{e}_y &= y_p - Q\hat{u}_D \\
\hat{u}_D &= \Theta \hat{e}_D
\end{align*}
\]  

(7)

In this paper, we will be interested in estimating and rejecting step disturbances of unknown amplitude which can be represented in the form of Eq. (5) and (6) as \( \phi_D \equiv 1 \), with \( F = 0, \Theta = 1 \), and \( L \) unknown. This has been a viable model for wind fluctuations in our previous work.

Our control objective in Disturbance Tracking Control will be to cause the output of the plant, \( y_p(t) \), to asymptotically track a linear function of the disturbance \( u_D \), which is not measured. We define the output tracking error vector as:

\[
e_y \equiv y_p - Q u_D
\]  

(8)

To achieve the desired control objective, we want \( e_y \to 0 \) as \( t \to \infty \). This aligns with the previous discussion for wind turbine Region II operation in Section 1.

Consider the plant given by Eq. (4) with the disturbance generator given by Eq. (5) and respective disturbance estimator given by (7). Our control objective for this system will be accomplished by an Adaptive Control Law of the form:

\[
u_p = G_e \hat{e}_y + G_D \hat{\phi}_D
\]  

(9)

where \( G_e \) and \( G_D \) are adaptive gain matrices of the appropriate compatible dimensions. We will need to use the estimated tracking error \( \hat{e}_y \) from Eq. 7 because the actual tracking error \( e_y \) is not measured. Now we specify the Adaptive Gain Laws, which will produce asymptotic tracking:
where \( \gamma_e, \gamma_D \) are arbitrary, positive definite matrices. Our Adaptive Controller is specified by Eq. (8) with the above adaptive gain laws Eq. (9).

We have the following **Stability and Convergence Theorems** which is proved in the Appendix:

**Theorem 1:**

a) \( CB > 0 \)

b) \( H(s)P(s) \) is minimum phase, where \( H(s) \equiv I - Q\theta(sI - F_c)^{-1}K_D; F_c \equiv F - K_DQ\theta \)

c) \( \phi_D \) bounded

d) \( \{ \text{the transmission zeros of } (A,B,C) \} \cap \{ \text{the eigenvalues of } F \} = \phi \)

\[ \Rightarrow \text{The adaptive control law: } u_p = G_e \hat{e}_y + G_D \phi_D \]

where

\[
\begin{align*}
\dot{G}_e &= -\hat{e}_y \hat{e}_y^T \gamma_e \\
\dot{G}_D &= -\hat{e}_y \phi_D^T \gamma_D
\end{align*}
\]

using the estimator:

\[
\begin{align*}
\dot{\hat{e}}_y &= y_p - Q\hat{u}_D \\
\dot{\hat{u}}_D &= \Phi \hat{e}_D
\end{align*}
\]

produces:

\[
\begin{align*}
\hat{e}_y &= y_p - Q\hat{u}_D = e_y - Q\theta e_D \quad \xrightarrow{t \to \infty} 0 \\
e_D &= \hat{z}_D - z_D \quad \xrightarrow{t \to \infty} 0
\end{align*}
\]

with bounded adaptive gains.

We will need the existence of so-called **Ideal Trajectories:**

\[
\begin{align*}
\dot{x}_* &= Ax_* + Bu_* + \Gamma u_D \\
y_* &= Cx_* = Qu_D
\end{align*}
\]

which is equivalent to the **Matrix Matching Conditions:**

\[
\exists! (S^*_1, S^*_2) \quad \begin{cases}
S^*_1F = AS^*_1 + BS^*_2 + \Gamma \theta \\
CS^*_1 = Q\theta
\end{cases}
\]

The following Lemma from [7] gives conditions for unique solvability:

**Lemma:** When \( CB \) is nonsingular,
Theorem 2:

When \( H(s)P(s) \) has a nonminimum phase zeros (as in the wind speed estimator to be used in next part) we can cancel out those zeros by using output feedback around the plant with appropriate choice of feedback constant \( K \). Let \( P_{FB}(s) \) be the new plant transfer function. We establish the new control law as follows:

- \( a) \ CB > 0 \)
- \( b) \ H(s)P_{FB}(s) \) is minimum phase, where \( H(s) = I - Q\theta(sI - F_c)^{-1}K_D; F_c \equiv F - K_DQ\theta; \)
- \( P_{FB}(s) = C(sI - (A + BK_C))^{-1}B \)
- \( c) \ \phi_D \) bounded
- \( d) \) the transmission zeros of \( (A, B, C) \) ∩ \( \{ \text{the eigenvalue s of } F \} = \phi \)

⇒ The adaptive control law : \( u_p \equiv Ky + G_e\hat{e}_y + G_D\phi_D \)

with

\[
\begin{align*}
\dot{G}_e &= -\hat{e}_y e_y^T y_e \\
\dot{G}_D &= -\hat{e}_y \phi_D^T y_D \\
\end{align*}
\]

using the estimator:

\[
\begin{align*}
\dot{\hat{e}}_y &\equiv y_p - Q\hat{u}_D \\
\dot{\hat{u}}_D &\equiv \theta\hat{e}_D \\
\end{align*}
\]

produces:

\[
\begin{align*}
e_y \xrightarrow{t \to \infty} 0 \\
\hat{e}_y &\equiv y_p - Q\hat{u}_D = e_y - Q\theta e_D \xrightarrow{t \to \infty} 0 \\
e_D &\equiv \hat{z}_D - z_D \xrightarrow{t \to \infty} 0 \\
\end{align*}
\]

with bounded adaptive gains.

III. Application to a Linearized Model of the National Renewable Energy Laboratory Controls Advanced Research Turbine

A follow-up goal outside of region III has been a desire to design a DTC (Disturbance Tracking Control) model in region II. This design will drive the error (a function of estimated wind speed and turbine rotor speed) to 0 as time approaches infinity. In region III, because the generator is already delivering maximum output power, it is not desired to adapt its operation, which is why the blade pitch is changed based upon the wind input disturbance. However, in region II, the generator is not delivering maximum power. With this in mind, it is able to develop a torque controller for the generator that is controlled by a tip speed ratio from the blades on the turbine. It is the objective of this controller to maximize power output and efficiency in operating region II of the turbine.

The turbine for modeling is called the Controls Advanced Research Turbine (CART) used in NREL (National Renewable Energy Laboratory). It is a 600 kW upwind machine with two-bladed, teetered rotor has a radius of \( R=21.3 \) m. Control inputs include generator torque and individual blade pitch. In region II, we keep the blade pitch angle as a constant, and only control generator torque.
We linearized the CART using constant wind speed of 8 m/s to ensure that the operating point lie in region II. This linearization gives the three state model of CART with parameters:

\[ A = \begin{pmatrix} 0 & 0 & 1 \\ 419.8 & 0 & 0 \\ -503.5 & -0.0330 & 0.0330 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -0.6735e^{-3} \\ 0.6735e^{-3} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 412.2 & 0 \end{pmatrix} \text{ and } \Gamma_D = \begin{pmatrix} 0 \\ 0 \\ 0.0511 \end{pmatrix} \]

This is equivalent to transfer function:

\[ P(s) = \frac{-0.2766s^2 - 0.009168s - 23.24}{s^3 + 0.03303s^2 + 503.3s + 13.86} \]

Since we want to reject step disturbance, the disturbance estimator will have the form:

\[ \dot{\hat{z}}_D = 0.5\dot{z}_D + K_D\hat{\theta}_y \]

\[ u_D = 1.0\hat{z}_D \]

Where \( F = 0 \), and \( \Theta = 1 \)

With estimator transfer function

\[ P_{est} = \frac{K_DQ\Theta}{S + K_DQ\Theta} \]
The interior of the torque controller is shown in Figure 2. We can change the gain $\gamma_c$ and $\gamma_D$ to adjust the performance of output just like the overshoot and Settling Time. The actual values of the gains used here in the adaptive controller are: $\gamma_c = 0.05; \gamma_D = 250$

Interior of the Disturbance Estimator is shown in Figure 3. We can change the value of $K_D$ to match the performance output as well as the wind profile. The actual value of $K_D$ used in this particular model is 0.683.

For comparison, we run the simulation with estimated wind using disturbance estimator, as well as actual wind. Figure 4 below is a plot of the wind file we used as a disturbance for simulations on both cases in order to obtain consistency with our test runs. To test this design we will first test on step wind, then on turbulent wind.

A. Step wind
Figure 4. Step Wind Used in Simulation

Figure 4 is a plot of a wind input file as a function of time that is step wind only in Region II. We use this wind profile to run Adaptive Disturbance Tracking Controller using disturbance estimator as well as actual wind and compare the results.

Figure 5. Estimated Wind Speed Compared with Actual Wind Speed
Figure 5 shows the estimated wind speed for step wind. It is seen that estimated wind speed is nearly equal to actual speed with some offset at lower wind speed but it is equal to actual wind speed at steady state as desired in theorem.

![Figure 5: Estimated vs Actual Wind Speed](image1)

**Figure 6. Tip Speed Ratio Comparison**

Figure 6 shows the comparison between Tip Speed Ratio (TSR) using wind speed from estimator and actual wind speed in controller. It is seen that the TSR is nearly same in both cases. For lower wind speed there is small offset but there is no offset at steady state. This is also the requirement for validation of the theorem developed before.

B. Turbulent wind

![Figure B: Turbulent Wind Speed](image2)
Figure 7. Turbulent Wind inflow used in simulation

Figure 7 is a plot of a turbulent wind input file in Region II. We compare actual wind speed with estimated wind speed, and TSR with actual wind speed and estimated wind speed.

![Figure 7](image1)

Figure 8. Wind Speed Comparison for Turbulent Wind

Figure 8 shows the turbulent wind speed profile and output from wind speed estimator. Estimated wind speed is close enough to the actual wind speed.

![Figure 8](image2)
Figure 14 shows the TSR comparison for turbulent wind using actual wind speed and wind speed from disturbance estimator. It is seen that the TSR is close and almost constant.

IV. Conclusion
In this paper the disturbance estimator was augmented in the theory of Adaptive Disturbance Tracking Control [1] which extends the fixed gain DTC theory of [3]-[4]. Results obtained from the linearized CART model shows that Adaptive Controller with estimated wind speed has same performance as with actual wind speed in both the setp wind and turbulent wind profile. Also, It is easy to see that the adaptive version of DTC in (8)-(9) is vastly simpler than the fixed gain version.

Numerical results on a simulation of the Linearized CART Turbine show the promise of adaptive DTC in tracking a desired Tip Speed Ratio in Region II operation.

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References
Appendix:

Proof of Stability and Convergence Theorem I

From hypothesis (d), the ideal trajectories exist.
\[
\begin{align*}
\Delta x &\equiv x - x_* \\
\Delta u &\equiv u - u_* \\
\Delta y &\equiv y - y_* = y - Qu_D \equiv e_y
\end{align*}
\]

Let
\[
\begin{align*}
\Delta x &= A\Delta x + B\Delta u \\
\Delta y &= C\Delta x = e_y
\end{align*}
\]

Define \( e_D \equiv \hat{z}_D - z_D \Rightarrow \hat{e}_D = Fe_D + K_D (y - Qu_D) = (F - K_D Q \theta)e_D + K_D (y - Qu_D) \)
\[
\begin{cases}
\hat{e}_D = F_c e_D + K_D e_y \\
F_c = F - K_D Q \theta \text{ stable}
\end{cases}
\]

Then the adaptive control law becomes
\[
\begin{align*}
u &\equiv G_e \hat{e}_y + G_D \phi_D = G_e \hat{e}_y + G_D \phi_D + \Delta G \eta \\
\Rightarrow \Delta u &\equiv u - u_* = (G_e \hat{e}_y + G_D \phi_D + \Delta G \eta) - (S^*_D \phi_D) \\
&= G_e \hat{e}_y + (G_D - S^*_D \phi_D) + w \\
&= G_e \hat{e}_y + w
\end{align*}
\]

with \( G_D^* \equiv S^*_D \), \( w \equiv \Delta G \eta, \Delta G \equiv [\Delta G_e \quad \Delta G_D] \) and \( \eta \equiv \left[ \begin{array}{c} \hat{e}_y \\ \phi_D \end{array} \right] \).

Now, let \( \xi \equiv \left[ \begin{array}{c} \Delta x \\ e_D \end{array} \right] \) then
\[
\begin{align*}
\dot{\xi} &= \overline{A}_c \xi + \overline{B} w \\
\dot{e}_y &= \Delta y = C\Delta x \\
\Rightarrow \hat{e}_y &\equiv y - Qu_D = y - Qu_D - Q \theta e_D = e_y - Q \theta e_D = C\Delta x - Q \theta e_D = \left[ \begin{array}{c} C \\ -Q \theta \end{array} \right] \xi
\end{align*}
\]

where
\[
\overline{A}_c \equiv \overline{A} + \overline{B} G_e^* \overline{C} = \left[ \begin{array}{cc} A + BG_e^* C & -BG_e^* Q \theta \\ K_D C & F_c \end{array} \right]
\]
and
\[
\overline{A} \equiv \left[ \begin{array}{cc} A & 0 \\ K_D C & F_c \end{array} \right], \overline{B} \equiv \left[ \begin{array}{c} B \\ 0 \end{array} \right], \overline{C} \equiv \left[ \begin{array}{c} C \\ -Q \theta \end{array} \right]
\]

Note that \( \overline{C} \overline{B} = CB \)
and \( P(s) \equiv \mathcal{C}(sI - A)^{-1}B = [C \quad -Q\theta \begin{bmatrix} sI - A & 0 \\ -K_\theta C & sI - F_C \end{bmatrix}^{-1}B] \)

\[
= \begin{bmatrix} C & -Q\theta \end{bmatrix} \begin{bmatrix} (sI - A)^{-1} \\ (sI - F_C)^{-1} \end{bmatrix} \mathcal{C}(sI - A)^{-1}B = \begin{bmatrix} I - Q\theta(sI - F_C)^{-1}K_\theta \end{bmatrix}P(s) \text{ is minimum phase.}
\]

\( \therefore (A, B, \mathcal{C}) \text{ASPR } \iff CB > 0 \text{ and } [I - Q\theta(sI - F_C)^{-1}K_\theta]P(s) \text{ is minimum phase. When } (A, B, C) \text{ is ASPR, we have } CB > 0 \text{ and } P(s) \text{ minimum phase. But the product } H(s)P(s) \text{ may not be minimum phase.}
\]

But, we will suppose it is in (b), then (A, B, \mathcal{C}) is ASPR and \( \exists \bar{P}, \bar{Q} > 0 \text{ with } \begin{bmatrix} \bar{A}_c^T\bar{P} + \bar{P}\bar{A}_c = -\bar{Q} \\
\bar{P}\bar{B} = \bar{C}^T
\end{bmatrix} \).

We consider the stability analysis next.

Let \( \begin{cases}
\dot{\xi} = \Delta\dot{\eta} \equiv -\dot{\hat{\eta}}^T\gamma \\
\dot{\hat{\eta}} \equiv y - Q\hat{u}_\theta
\end{cases} \)

Define \( V(\Delta G) \equiv \frac{1}{2}tr(\Delta G\gamma^{-1}\Delta G^T) \rightarrow \dot{V}(\Delta G) = tr(\Delta\dot{G}\gamma^{-1}\Delta G^T) = -tr(\dot{\hat{\eta}}^T\gamma\gamma^{-1}\Delta G^T) \)

\( = -tr(\dot{\hat{\eta}}^Tw^T) = -\langle \dot{\hat{\eta}} , w \rangle \)

Also \( V(\xi) \equiv \frac{1}{2}\xi^T\bar{P}\xi \) with \( \bar{P} > 0 \) \( \exists \begin{bmatrix} \bar{A}_c^T\bar{P} + \bar{P}\bar{A}_c = -\bar{Q} < 0 \\
\bar{P}\bar{B} = \bar{C}^T
\end{bmatrix} \)

\( \rightarrow \dot{V}(\xi) = \xi^T\bar{P}\dot{\xi} = \xi^T\bar{P}[\bar{A}_c\xi + \bar{B}w] = -\frac{1}{2}\xi^T\bar{Q}\xi + \langle \bar{C}\xi , w \rangle
\)

\( = -\frac{1}{2}\xi^T\bar{Q}\xi + \langle C - Q\theta \begin{bmatrix} \Delta x \\ e_D \end{bmatrix}^T, w \rangle = -\frac{1}{2}\xi^T\bar{Q}\xi + \langle \dot{\hat{\eta}} , w \rangle \)

Form: \( V(\xi, \Delta G) \equiv V(\xi) + V(\Delta G) \)

\( \rightarrow \dot{V}(\xi, \Delta G) = -\frac{1}{2}\xi^T\bar{Q}\xi + \langle \dot{\hat{\eta}} , w \rangle - \langle \dot{\hat{\eta}} , w \rangle = -\frac{1}{2}\xi^T\bar{Q}\xi \leq 0
\)

\( \therefore (\xi, \Delta G) \text{ is bounded.}
\]

Also, \( \dot{V}(\xi, \Delta G) = -\xi^T\bar{Q}\dot{\xi} = -\xi^T\bar{Q}[\bar{A}_c\xi + \bar{B}\Delta G\eta] \) is bounded when \( \phi_0 \) is bounded as assumed in (c).

From Barbalat’s lemma, \( \dot{\xi} = \begin{bmatrix} \Delta x \\ e_D \end{bmatrix} \rightarrow 0 \) as \( t \rightarrow \infty \). Therefore
\[ \begin{cases} e_y = \Delta y = C\Delta x \rightarrow 0 \\ \dot{e}_y \equiv y - Q\hat{u}_D = e_y - Q\hat{\theta}e_D \rightarrow 0 \end{cases} \]

Proof of Stability and Convergence Theorem II

From hypothesis (d), the ideal trajectories exist.

Let
\[ \Delta x \equiv x - x_* \]
\[ \Delta u \equiv u - u_* \]
\[ \Delta y \equiv y - y_* = y - Qu_D \equiv e_y \]

Define \( e_D \equiv \hat{z}_D - z_D \Rightarrow \dot{e}_D = F_eD + K_D(y - Qu_D) \Rightarrow (F - K_DQ\hat{\theta})e_D + K_D(y - Qu_D) \]
\[ \dot{e}_D = F_eD + K_De_y \]
\[ \therefore F_c \equiv F - K_DQ\hat{\theta} \text{ stable} \]

Then the adaptive control law becomes
\[ u \equiv Ky + G^*_\epsilon \hat{\epsilon}_y + G^*_D \phi_D = Ky + G^*_\epsilon \hat{\epsilon}_y + G^*_D \phi_D + \Delta G \eta \]
\[ u \equiv K(\hat{\epsilon}_y + Qu_D) + G^*_\epsilon \hat{\epsilon}_y + G^*_D \phi_D + \Delta G \eta \]
\[ u \equiv K(\hat{\epsilon}_y + Qu_D + \theta e_D) + G^*_\epsilon \hat{\epsilon}_y + G^*_D \phi_D + \Delta G \eta \]
\[ u \equiv (K + G^*_\epsilon)\hat{\epsilon}_y + KQ\theta e_D + (KQL + G^*_D)\phi_D + \Delta G \eta \]
\[ \Rightarrow \Delta u = u - u_* = [(K + G^*_\epsilon)\hat{\epsilon}_y + KQ\theta e_D + (KQL + G^*_D)\phi_D + \Delta G \eta] - (S^*_ZL\phi_D) \]
\[ = (K + G^*_\epsilon)\hat{\epsilon}_y + KQ\theta e_D + (KQL + G^*_D - S^*_ZL)\phi_D + w \]
\[ = (K + G^*_\epsilon)\hat{\epsilon}_y + KQ\theta e_D + w \]

with \( G^*_D \equiv S^*_ZL - KQL, w \equiv \Delta G \eta, \Delta G \equiv \left[ \Delta G_e \quad \Delta G_D \right] \text{ and } \eta \equiv \left[ \hat{\epsilon}_y \right] \).

\[ \Delta \dot{x} = A\Delta x + B\Delta u = A\Delta x + B((K + G^*_\epsilon)\hat{\epsilon}_y + KQ\theta e_D + w) \]
\[ \Delta \dot{x} = A\Delta x + B((K + G^*_\epsilon)(C\Delta x - Q\theta e_D) + KQ\theta e_D + w) \]
\[ \Rightarrow \Delta \dot{x} = (A + BKC + BG^*_C)\Delta x - B(KQ\theta e_D - G^*_\epsilon Q\theta e_D + KQ\theta e_D + w) \]
\[ \Delta \dot{x} = (A + BKC + BG^*_C)\Delta x - BG^*_\epsilon Q\theta e_D + Bw \]
\[ \Delta y = C\Delta x = e_y \]

Now, let \( \xi \equiv \begin{bmatrix} \Delta x \\ e_D \end{bmatrix} \)

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\[
\begin{aligned}
\hat{\xi} &= \bar{A}_c \xi + \bar{B}_w \\
\hat{e}_y &= \Delta y = CAx \\
\Rightarrow \hat{\xi} \\&= \frac{1}{\xi} \left( y - Qu_D \right) = \frac{y - Qu_D - Q\hat{e}_D = e_y, \hat{e}_D = C\Delta x - Q\hat{e}_D = \left[ C - Q\theta \right] \hat{\xi} \right)
\end{aligned}
\]

where

\[
\bar{A}_c = \bar{A} + \bar{B}G^*_c \bar{C} = \begin{bmatrix} A + BKC + BG^*_c C & -BG^*_c Q\theta \\ K_D C & F_c \end{bmatrix}
\]

and

\[
\bar{A}_i = \begin{bmatrix} A + BKC & 0 \\ K_D C & F_c \end{bmatrix}, \bar{C} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} C & -Q\theta \end{bmatrix}
\]

Note that \( \bar{C}B = CB \)

\[
\begin{aligned}
\bar{P}(s) &= \bar{C}(sI - \bar{A}_i)^{-1} \bar{B} = \begin{bmatrix} C & -Q\theta \end{bmatrix} \begin{bmatrix} sI - (A + BKC) & 0 \\ -K_D C & sI - F_c \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} C & -Q\theta \end{bmatrix} \begin{bmatrix} (sI - (A + BKC))^{-1} & 0 \\ (sI - F_c)^{-1} K_D C (sI - (A + BKC))^{-1} & (sI - F_c)^{-1} 0 \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix}
\end{aligned}
\]

\[
I - Q\theta(sI - F_c)^{-1} K_D \bar{P}_{fb}(s) \text{ is minimum phase.}
\]

Where \( \bar{P}_{fb}(s) = C(sI - (A + BKC))^{-1} B \) is plant transfer function after output feedback is used.

\( \therefore (\bar{A}, \bar{B}, \bar{C}) \text{ ASPR } \Leftrightarrow CB > 0 \) and \( I - Q\theta(sI - F_c)^{-1} K_D \bar{P}_{fb}(s) \text{ is minimum phase. When } (A, B, C) \text{ is ASPR, we have } CB > 0 \) and \( P(s) \text{ minimum phase. But the product } H(s)P(s) \text{ may not be minimum phase. In such case we can set the value of } K \text{ such that } H(s)P_{fb}(s) \text{ is minimum phase.}
\]

then \( (\bar{A}, \bar{B}, \bar{C}) \text{ is ASPR and } \exists \bar{P}, \bar{Q} > 0 \text{ } \exists \bar{A}_c \bar{P} + \bar{P}\bar{A}_c = -\bar{Q} \text{ and } \bar{P}\bar{B} = \bar{C}^T \)

We consider the stability analysis next.

Let

\[
\begin{aligned}
\dot{G} &= \Delta \dot{G} \equiv -\hat{e}_y \eta^T \gamma \\
\hat{\xi} &\equiv y - Qu_D \\
\end{aligned}
\]

define

\[
V(\Delta G) \equiv \frac{1}{2} tr(\Delta G \gamma^{-1} \Delta G^T) \Rightarrow \dot{V}(\Delta G) = tr(\Delta \dot{G} \gamma^{-1} \Delta G^T) = -tr(\hat{e}_y \eta^T \gamma^{-1} \Delta G^T) = -tr(\hat{e}_y \gamma^{-1} \Delta \dot{G}^T)
\]

\[
= -tr(\hat{\xi} w^T) = -\frac{1}{2} \hat{\xi}^T \bar{P} \hat{\xi}
\]

Also \( V(\xi) = \frac{1}{2} \xi^T \bar{P} \xi \text{ with } \bar{P} > 0 \text{ } \exists \bar{A}_c \bar{P} + \bar{P}\bar{A}_c = -\bar{Q} < 0 \text{ and } \bar{P}\bar{B} = \bar{C}^T \)
\[ \dot{V}(\xi) = \dot{\xi}^T \overline{F} \dot{\xi} = \dot{\xi}^T [\overline{A}_c \xi + \overline{B} w] = -\frac{1}{2} \dot{\xi}^T \overline{Q} \dot{\xi} + \langle \overline{C} \dot{\xi}, w \rangle \]

\[ = -\frac{1}{2} \dot{\xi}^T \overline{Q} \dot{\xi} + \left[ C - Q \theta \right] \frac{\Delta x}{e_D}, w \right] = -\frac{1}{2} \dot{\xi}^T \overline{Q} \dot{\xi} + \langle \dot{e}_y, w \rangle \]

Form: \( V(\xi, \Delta G) \equiv V(\xi) + V(\Delta G) \)

\[ \dot{V}(\xi, \Delta G) = -\frac{1}{2} \dot{\xi}^T \overline{Q} \dot{\xi} + \langle \dot{e}_y, w \rangle - \langle \dot{e}_y, w \rangle = -\frac{1}{2} \dot{\xi}^T \overline{Q} \dot{\xi} \leq 0 \]

\[ \therefore (\xi, \Delta G) \text{ is bounded.} \]

Also, \( \ddot{V}(\xi, \Delta G) = -\dot{\xi}^T \overline{Q} \dot{\xi} = -\dot{\xi}^T \overline{Q}[\overline{A}_c \xi + \overline{B} \Delta G \eta] \) is bounded when \( \phi_0 \) is bounded as assumed in (c).

From Barbalat's lemma, \( \dot{\xi} \equiv \left[ \frac{\Delta x}{e_D} \right] \rightarrow 0 \). Therefore

\[ \begin{cases} 
\dot{e}_y = \Delta y = C \Delta x \rightarrow 0 \\
\dot{e}_y \equiv y - Q \dot{u}_D = e_y - Q \theta e_D \rightarrow 0
\end{cases} \]

This completes the proof.