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Inverse heat conduction problems with boundary and final time measured output data

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This article presents a systematic study of inverse problems of identifying the unknown source term \( F(x, t) \) of the heat conduction (or linear parabolic) equation

\[
\frac{\partial u}{\partial t} = (k(x)\frac{\partial u}{\partial x}) + F(x, t), \quad (x, t) \in \Omega_T,
\]

\[
u(x, 0) = 0, \quad x \in (0, l),
\]

\[
u_x(0, t) = 0, \quad u(l, t) = 0, \quad t \in (0, T],
\]

\[
h(t) := u(0, t), \quad t \in (0, T].
\]

where \( k(x) \) is the thermal conductivity coefficient, \( u \) is the temperature, \( F \) is the unknown source term, \( h \) is the final time observation. In the first part of this article the adjoint problem approach is used to derive formulas for the Fréchet gradient of cost functionals via solutions of the corresponding adjoint problems. It is proved that all these gradients are Lipschitz continuous. A necessary conditions for unicity and hence distinguishability of solutions of all the three types of inverse source problems are derived. In the second part of this article the semigroup theory is used to obtain a general representation of a solution of the inverse source problem for the abstract evolution equation \( u_t = Au + F \) with final data overdetermination. This representation shows non-uniqueness structure of the inverse problem solution, and also permits one to derive a sufficient condition for unicity.

Keywords: inverse source problem; boundary/final data observation; Fréchet gradient; Lipschitz continuity; semigroup representation; unicity; Fourier method

AMS Subject Classifications: 35R30; 35K05; 49N45; 47A05

1. Introduction

We study the following typical inverse problems for determining the unknown source term \( F(x, t) \) from measured output data in the form of additional Dirichlet or Neumann-types of boundary conditions, or final time overdetermination.

The first inverse source problem is defined as

\[
\begin{align*}
\frac{\partial u}{\partial t} &= (k(x)\frac{\partial u}{\partial x}) + F(x, t), \quad (x, t) \in \Omega_T, \\
u(x, 0) &= 0, \quad x \in (0, l), \\
u_x(0, t) &= 0, \quad u(l, t) = 0, \quad t \in (0, T], \\
h(t) &= u(0, t), \quad t \in (0, T].
\end{align*}
\]
given in the parabolic domain $\Omega_T:=\{0 < x < l, \ 0 < t \leq T\}$. The function $h(t)$, with $h(0)=0$, is defined to be the Dirichlet-type output measured data, given as the measured temperature at the left end of a bar occupied at the interval $[0,l]$. The inverse source problem (1)–(2) will be referred as ISP1.

The second inverse problem is defined as

$$
\begin{align*}
\begin{cases}
  u_t = (k(x)u_x)_x + F(x,t), & (x,t) \in \Omega_T, \\
  u(x,0) = F(x), & x \in (0,l), \\
  u(0,t) = 0, & u_x(l,t) = 0, & t \in (0,T],
\end{cases}
\end{align*}
$$

(3)

$$
f(t) := -k(0)u_x(0,t), \ t \in (0,T].
$$

(4)

The function $f(t)$, with $f(0)=0$, is defined to be the Neumann-type output measured data, given as the measured flux at the left end of a bar. The inverse source problem (3)–(4) will be referred as ISP1.

The third inverse problem is defined as

$$
\begin{align*}
\begin{cases}
  u_t = (k(x)u_x)_x + F(x,t), & (x,t) \in \Omega_T, \\
  u(x,0) = u_0(x), & x \in (0,l), \\
  u(0,t) = 0, & u_x(l,t) = 0, & t \in (0,T],
\end{cases}
\end{align*}
$$

(5)

$$
u_T(x) := u(x,T), \ x \in [0,l].
$$

(6)

The function $u_T(x)$, with $u_T(0)=u'_T(l)=0$, is defined to be the final time overdetermination, given as the measured temperature at the final time $t=T$.

For a given function $F \in \mathcal{F}$, from the set of admissible coefficients $\mathcal{F}$, the parabolic problems (1), (3) and (5) will be referred as direct (or forward) problems, corresponding to the inverse source problems ISP1, ISP2 and ISP3, respectively. A solution of a direct problem, corresponding to the given function $F \in \mathcal{F}$, will be defined as $u(x,T;F)$. The functions $h(t)$, $f(t)$ and $u_T(x)$ are defined as the measured output data.

This article is concerned with the mathematical analysis of the above-defined inverse source problems, from the points of view of adjoint problem approach [1], semigroup theory [2,3] and Fourier method. Most typical mathematical models related to identification of the unknown source term $F(x,t)$ in parabolic equations are governed by ISP1, ISP2 or ISP3 (see [4–8], and references therein). The inverse source problems for linear parabolic equations with measured output data in the form of the Dirichlet/Neumann-types of boundary conditions, of final time overdetermination have been considered by many authors (see, for instance, [1,9–21]). Thus, using the spectral theory, an existence and uniqueness of the solution of an inverse source problem for the case $F=F(x)$ are derived in [10]. Global uniqueness for ISP3 has been considered in [9], for ISP1 with state-dependent source term $F(u)$ has been considered in [11]. An existence result for the parabolic equation $u_t=\Delta u + p(x)u + F(u)$ has been proved in [14] under the assumption of convexity of the space domain $\Omega$. Determination of the unknown function $p(x)$ in the source term $F=p(x)f(u)$ in the parabolic equation $u_t-\Delta u + p(x)f(u)$ from the overspecified data (2) by using a fixed point theory, has been proposed in [21].

Note that in all the above-cited works one or another aspects of a given inverse source problem are studied. In the proposed study, most distinguished features of all the above-formulated typical inverse source problems are analysed. Specifically, the adjoint problem
approach based on the weak solution theory for PDE, and subsequent use of quasisolution approach is applied here to ISP1, ISP2 and ISP3. This approach permits one to prove Fréchet differentiability of the corresponding cost functionals

\[
\begin{align*}
J_1(F) &= \int_0^T [u(0, t; F) - h(t)]^2 \, dt, \\
J_2(F) &= \int_0^T [k(0)u_x(0, t; F) + f(t)]^2 \, dt, \\
J_3(F) &= \int_0^1 [u(x, T; F) - u_T(t)]^2 \, dx,
\end{align*}
\]

and derive explicit formulas for the Fréchet gradients of these functionals via the solutions of the corresponding well-posed adjoint problems. Then the Lipschitz continuity of the gradients of these functionals are proved. These results are then used to construct gradient-based iteration algorithm for numerical reconstruction of the unknown source term. On the other hand, convexity condition obtained for the cost functionals provides further insight into the uniqueness of inverse source problems. The strong unicity result for ISP3 is derived by using the semigroup theory for abstract parabolic equations in the second part of this article. Analysing semigroup property of the input–output map, a general representation of a solution of ISP3 is derived. This representation allows us to derive the non-uniqueness structure of a solution. As an application, Fourier method is employed for ISP3 to derive an explicit formula for unique solution of the inverse source problem with time-independent source term \( F(x) \).

This article is organized as follows. In Section 2 the adjoint problem approach is introduced for ISP1. An explicit relationship between the weak solution of ISP1 and the corresponding adjoint problem obtained here allows us to derive a formula for the gradient of the cost functional \( J_1(F) \). Then, Fréchet differentiability of this functional \( J_1(F) \) is proved. Lipschitz continuity of the gradient \( J_1(F) \) and convergence of the gradient method is obtained in Section 3. For ISP2 and ISP3 similar results are derived in Section 4 and Section 5, correspondingly. Convexity of the cost functionals, corresponding to the considered inverse source problems, and unicity of solutions of these problems are considered in Section 6. In Section 7 the semigroup approach is proposed for the inverse source problem with final data overdetermination for abstract evolution equation. An illustration of the unicity result an analysis of the inverse source problem with final data overdetermination by Fourier method is presented. Some concluding remarks are given in Section 8.

2. Adjoint problem approach for ISP1: Fréchet differentiability of the cost functional

Let \( F \subset H^0(\Omega_T) \) be a closed convex set of admissible unknown sources. With respect to the coefficient \( k(x) > 0 \) and the output data \( h(t) \), we will assume that

\[
k(x) \in L_\infty[0, 1], \quad k^* \geq k(x) \geq k_* > 0, \quad h(t) \in H^0[0, T].
\]

The weak solution of the direct problem (1) will be defined to be as the function \( u \in \tilde{V}^{1,0}(\Omega_T) \), which satisfies the following integral identity [7,22]:

\[
\int_0^1 \int_{\Omega_T} (-u v_t + ku_x v_x) \, dx \, dt = \int_0^1 \int_{\Omega_T} F(x, t)v(x, t) \, dx \, dt, \quad \forall v \in \tilde{H}^{1,1}(\Omega_T),
\]
with \( v(x, T) = 0 \). Here \( V^{1,0}(\Omega_T) := C([0, T]; H^0(0, l)) \cap H^0((0, T); H^1(0, l)) \) is the Banach space of functions with the norm

\[
\|u\|_{V^{1,0}(\Omega_T)} := \text{vrai max}_{t \in [0, T]} \|u\|_{H^0(0, l)} + \|u_x\|_{H^0(\Omega_T)},
\]

\( V^{1,0}(\Omega_T) := \{ v \in V^{1,0}(\Omega_T) : v(l, t) = 0, \forall t \in (0, T) \} \), \( H^{1,1}(\Omega_T) := \{ v \in H^{1,1}(\Omega_T) : v(l, t) = 0, \forall t \in (0, T) \} \) and \( H^{1,1}(\Omega_T) \) is the Sobolev space with the norm

\[
\|u\|_{H^{1,1}(\Omega_T)} := \left\{ \int \int_{\Omega_T} [u^2 + u_x^2 + u_t^2] \, dx \, dt \right\}^{1/2}.
\]

Note that the norms \( \|u\|_{H^0(0, l)} \) and \( \|u_x\|_{H^0(\Omega_T)} \) are equivalent due to the homogeneous Dirichlet condition \( u(l, t) = 0 \) in the direct problem (1).

Under the above conditions with respect to the given data, an existence and uniqueness of the weak solution \( u \in V^{1,0}(\Omega_T) \) of the parabolic problem (1) for general case \( u_x(0, t) = g(t), u(x, 0) = u_0(t) \) instead of \( u_x(0, t) = 0, u(x, 0) = 0 \), accordingly, and \( \Omega_T := \Omega \times (0, T) \), \( \Omega \in \mathbb{R}^n \), instead of \( \Omega_T := (0, l) \times (0, T) \) has been proved in [23]. The a priori estimate, obtained in [23] for this case, can be written as follows:

\[
\sup_{t \in [0, T]} \|u\|_{H^0(0, l)}^2 + \|u_x\|_{H^0(\Omega_T)}^2 \leq C \left[ \|u_0\|_{H^0(0, l)}^2 + \|F\|_{H^0(\Omega_T)}^2 + \|g\|_{H^0(0, T)}^2 \right].
\]  

Let us denote by \( u = u(x, t; F) \) the solution of the parabolic problem (1), corresponding to the given source term \( F \in \mathcal{F} \). If this function also satisfies the additional condition (2), then it must satisfy the non-linear functional equation \( u(0, t; F) = h(t), t \in (0, T) \). However, due to measurement errors, in practice the output data \( h(t) \) can only be given with some error (noise). Hence, the above exact equality cannot be fulfilled. For this reason we define a quasisolution of ISP1, according to [24,25], as a solution of the minimization problem for the cost functional \( J_1(F) \), given by (8):

\[
J_1(F_*) = \inf_{F \in \mathcal{F}} J_1(F).
\]

Evidently, if \( J_1(F_*) = 0 \), then the quasisolution \( F_* \in \mathcal{F} \) is also a strict solution of ISP1. On the other hand, if the sequence \( \{ F^{(n)} \} \subset \mathcal{F} \) converges to \( F \in \mathcal{F} \) in the norm of \( H^1(\Omega_T) \), then by estimate (10) the corresponding sequence of traces \( \{ (u(x, t; F^{(n)}))_{x=0} \} \) converges to \( \{ (u(x, t; F))_{x=0} \} \), in the norm of \( H^0(0, T) \). This means that \( J_1(F^{(n)}) \to J_1(F) \), as \( n \to \infty \), i.e. the functional \( J_1(F) \) is continuous on \( \mathcal{F} \). Then due to the Weierstrass’s existence theorem [21], the set of solutions \( \mathcal{F}_* := \{ w \in \mathcal{F} : J(w*) = J_* = \inf J(w) \} \) of the above minimization problem is not an empty set. This implies an existence of a quasisolution of ISP1.

Now let us assume that \( F, F + \Delta F \in \mathcal{F} \) and consider the first variation \( \Delta J_1(F) := J_1(F + \Delta F) - J_1(F) \) of the cost functional \( J_1(F) \). We have

\[
\Delta J_1(F) = 2 \int_0^T [u(0, t; F) - h(t)] \Delta h(t) \, dt + \int_0^T [\Delta u(0, t; F)]^2 \, dt,
\]  

(11)
where \( \Delta u(x, t; F) = u(x, t; F + \Delta F) - u(x, t; F) \in V^{1,0}(\Omega_T) \), \( \Delta h(t) = \Delta u(0, t; F) \), and the function \( \Delta u = \Delta u(x, t; F) \) is the solution of the following parabolic problem:

\[
\begin{align*}
\Delta u_t &= (k(x)\Delta u_x)_x + \Delta F(x, t), \quad (x, t) \in \Omega_T, \\
\Delta u(x, 0) &= 0, \quad x \in (0, l), \\
\Delta u_x(0, t) &= 0, \quad \Delta u(l, t) = 0, \quad t \in (0, T].
\end{align*}
\]

**Lemma 2.1** Let \( u(x, t; F), u(x, t; F + \Delta F) \in V^{1,0}(\Omega_T) \) be solutions of the direct problem (1) for the given sources \( F, F + \Delta F \in \mathcal{F} \). Suppose that \( h(t) := u(0, t; F), h(t) + \Delta h(t) = u(0, t; F + \Delta F) \) are the corresponding outputs. If \( \psi(x, t; F) \in V^{1,0}(\Omega_T) \) is the solution of the backward parabolic problem

\[
\begin{align*}
\psi_t &= -(k(x)\psi_x)_x, \quad (x, t) \in \Omega_T, \\
\psi(x, \tau) &= 0, \quad x \in (0, l), \\
k(0)\psi_x(0, t) &= p(t), \quad \psi(l, t) = 0, \quad t \in (0, \tau],
\end{align*}
\]

then for all \( \tau \in (0, T] \) the following integral identity holds:

\[
\int_{\Omega_T} \psi(x, t)\Delta F(x, t)\,dx\,dt = \int_0^\tau \Delta h(t) p(t)\,dt, \quad \forall p(t) \in H^0[0, T].
\]

**Proof** Multiplying both the sides of Equation (12) by the arbitrary function \( \psi(x, t) \), we get

\[
\int_{\Omega_T} \left[ \Delta u_t(x, t; F) - (k(x)\Delta u_x(x, t; F))_x \right] \psi(x, t)\,dx\,dt = \int_{\Omega_T} \Delta F(x, t)\psi(x, t)\,dx\,dt.
\]

Applying by parts integration formula to the left integral, we obtain

\[
\int_{\Omega_T} \left[ \Delta u_t(x, t; F) - (k(x)\Delta u_x(x, t; F))_x \right] \psi(x, t)\,dx\,dt \\
= -\int_{\Omega_T} \left[ \psi_t(x, t; F) + (k(x)\psi_x(x, t; F))_x \right] \Delta u(x, t; F)\,dx\,dt + \int_0^\tau \left[ \Delta u(x, t; F)\psi(x, t) \right]_{x=0}^{x=l}\,dx \\
- \int_0^\tau [k(x)\Delta u_x(x, t; F)\psi(x, t) - k(x)\Delta u(x, t; F)\psi_x(x, t)]_{x=0}^{x=l}\,dt.
\]

Now we require that the function \( \psi(x, t) \) is the solution of problem (13). Then the first right-hand side integral in (16) is identically equal to zero. Further, second right-hand side integral is also zero, due to the homogeneous initial \( \Delta u(x, 0; F) = 0 \) and final \( \psi(x, \tau) = 0 \) conditions. Finally, taking into account the boundary conditions in (12) and (13), with the output \( \Delta h(t) = \Delta u(0, t; F) \), in the last integral, we obtain

\[
\int_0^\tau [k(x)\Delta u_x(x, t; F)\psi(x, t) - k(x)\Delta u(x, t; F)\psi_x(x, t)]_{x=0}^{x=l}\,dt = -\int_0^\tau \Delta h(t) p(t)\,dt.
\]

This with (15) implies the proof.
The well-posed backward parabolic problem (13) is defined to be the adjoint problem, corresponding to ISP1.

Now let us choose the arbitrary control function \( p(t) \in H^0[0, T] \) in (13) as follows: \( p(t) = 2[u(0, t; F) - h(t)] \), where \( h(t) \) is the measured output data given by (2). Substituting this in the integral identity (14), we get

\[
\int \int_{\Omega_T} \Delta F(x, t) \psi(x, t) \, dx \, dt = 2 \int_0^T [u(0, t; F) - h(t)] \Delta h(t) \, dt.
\]

This, with the first variation formula (11), implies the following formal definition of the Fréchet gradient of the cost functional \( J_1(F) \):

\[
\Delta J_1(F) = \int \int_{\Omega_T} \psi(x, t) \Delta F(x, t) \, dx \, dt + \int_0^T [\Delta u(0, t; F)]^2 \, dt.
\] (17)

The following result asserts the mathematical framework of this definition.

**Lemma 2.2** Let the conditions given in (8) hold. Then for the solution \( \Delta u \in H^1,0(\Omega_T) \) of the parabolic problem (12), corresponding to a given source term \( \Delta F \in \mathcal{F} \), the following estimates hold:

\[
\|\Delta u\|^2_{H^1(\Omega_T)} \leq \frac{\varepsilon}{2\sigma_e} \|\Delta F\|^2_{H^0(\Omega_T)}, \quad \|\Delta u_x\|^2_{H^1(\Omega_T)} \leq \frac{\varepsilon}{F} \|\Delta F\|^2_{H^0(\Omega_T)},
\] (18)

where \( \sigma_e = 2k^*/\ell - 1/(2\varepsilon) \), \( \varepsilon > \ell^2/(4k^*) \).

**Proof** Multiplying both sides of the parabolic equation (12) by \( \Delta u \), integrating on \( \Omega_T \) and using the initial and boundary conditions, we obtain the following energy identity:

\[
\int \int_{\Omega_T} k(x)[\Delta u_x(x, t; F)]^2 \, dx \, dt + \frac{1}{2} \int_0^T [\Delta u(x, T; F)]^2 \, dx = \int \int_{\Omega_T} \Delta F(x, t) \Delta u(x, t; F) \, dx \, dt.
\]

Using here the Cauchy \( \varepsilon \)-inequality \( \alpha \beta \leq \varepsilon \alpha^2/2 + \beta^2/(2\varepsilon) \), \( \forall \alpha, \beta \in \mathbb{R}, \forall \varepsilon > 0 \), we get

\[
k_e \|\Delta u_x\|^2_{H^1(\Omega_T)} \leq \frac{\varepsilon}{2} \|\Delta F\|^2_{H^0(\Omega_T)} + \frac{1}{2\varepsilon} \|\Delta u\|^2_{H^0(\Omega_T)}.
\] (19)

To obtain the estimates given in (18), one needs to use the Poincaré inequality

\[
\|\Delta u_x\|^2_{H^1(\Omega_T)} \geq (2/\ell^2) \|\Delta u\|^2_{H^0(\Omega_T)}
\]
on the left-, and then, on the right-hand sides of (19). ■

**Corollary 2.1** Let the conditions of Lemma 2.2 hold. Then the last integral in (17) can be estimated as follows:

\[
\|\Delta u(0, \cdot; F)\|^2_{H^0[0, T]} := \int_0^T [\Delta u(0, t; F)]^2 \, dt \leq \gamma_\varepsilon \|\Delta F\|^2_{H^0(\Omega_T)}, \quad \gamma_\varepsilon = \frac{\varepsilon}{2\sigma_e} > 0.
\] (20)

The proof follows from the inequality \( \|\Delta u(0, \cdot; F)\|^2_{H^0[0, T]} \leq \ell \|\Delta u_x\|^2_{H^1(\Omega_T)} \) and the second estimate (18).

This corollary with the definition \( J_1(F + \Delta F) - J_1(F) = (J_1'(F), \Delta F)_{H^0(\Omega_T)} + o(\|\Delta F\|^2_{H^0(\Omega_T)}) \) of Fréchet differential implies the following theorem.

**Theorem 2.1** Let the conditions given in (8) hold and \( \mathcal{F} \subset H^0(\Omega_T) \). Then the cost functional \( J_1(F) \) defined by (7) is Fréchet-differentiable, \( J_1(F) \in C^1(\mathcal{F}) \). Moreover, Fréchet derivative at
Due to (21). Multiplying both the sides of Equation (25) by 3. Lipschitz continuity of the gradient functional can be determined via the well-posed adjoint problem (13), with the input data $p(t) = 2[u(0, t; F) - h(t)]$, which contains the measured output data $h(t)$. This result, with the gradient formula (22), suggests a use of gradient-type iterative methods for approximate solution of ISP1. However, any gradient method for the minimization problem requires an estimation of the iteration parameter $\alpha_n > 0$ in the iteration process

$$F^{(n+1)} = F^{(n)} - \alpha_n J'_1(F^{(n)}), \quad n = 0, 1, 2, \ldots,$$

with the given initial iteration $F^{(0)} \in F$. In the case of Lipschitz continuity of the gradient $J'_1(F)$, the parameter $\alpha_n$ can be estimated via the Lipschitz constant $L_1 > 0$, i.e. $0 < \delta_0 \leq \alpha_n \leq 2/(L_1 + 2\delta_1)$. Here $\delta_0, \delta_1 > 0$ are arbitrary parameters. The following result shows the continuity of the gradient $J'_1(F)$.

**Lemma 3.1** Let the conditions of Theorem 2.1 hold. Then the functional $J_1(F)$ is of Hölder class $C^{1,1}(F)$ and

$$\|J'_1(F + \Delta F) - J'_1(F)\|_{F^1(\Omega_T)} \leq L_1 \|\Delta F\|_{F^1(\Omega_T)}, \quad \forall F, F + \Delta F \in F,$$

where $L_1 = (2\epsilon_1\gamma\sigma_l/\sigma_k)^{1/2}l > 0$ is the Lipschitz constant, $\sigma_k = k_* - l/\epsilon_1 > 0$, the parameter $\gamma > 0$ is defined in Corollary 2.1, and the arbitrary parameter $\epsilon_1 > 0$ satisfies the condition: $\epsilon_1 > l/k_*$. 

**Proof** By the gradient formula (22) we have:

$$\|J'_1(F + \Delta F) - J'_1(F)\|_{F^1(\Omega_T)} := \|\Delta \psi(\cdot, t; F)\|_{F^1(\Omega_T)}, \quad \text{where the function } \Delta \psi(x, t; F) := \psi(x, t; F + \Delta F) - \psi(x, t; F) \in V^{1,0}(\Omega_T)$$

is the solution of the following backward parabolic problem:

$$\begin{cases}
\Delta \psi_t = -(k(x)d)\sigma_k, & (x, t) \in \Omega_T, \\
\Delta \psi(x, T) = 0, & x \in (0, l), \\
-k(0)\Delta \psi_x(0, t) = 2\Delta u(0, t; F), \quad \Delta \psi(l, t) = 0, & t \in (0, T],
\end{cases}$$

due to (21). Multiplying both the sides of Equation (25) by $\Delta \psi(x, t; F)$, integrating on $\Omega_T$ and using the initial and boundary conditions, we obtain the following energy identity:

$$\int \int_{\Omega_T} k(x)[\Delta \psi_x(x, t; F)]^2 dx dt + \frac{1}{2} \int_0^T [\Delta \psi(x, 0; F)]^2 dx = 2 \int_0^T \Delta u(0, t; F)\Delta \psi(0, t; F) dt.$$
Applying the Cauchy $\varepsilon$-inequality to the right-hand side integral, we conclude that

$$k_s \int_{\Omega_T} [\Delta \psi(x, t; F)]^2 \, dx \, dt \leq \varepsilon_1 \int_0^T [\Delta u(0, t; F)]^2 \, dt + \frac{1}{\varepsilon_1} \int_0^T [\Delta \psi^2(0, t; F)] \, dt,$$

Here using Corollary 2.1 and the inequality \( \|\Delta \psi(0, \cdot; F)\|^2_{L^2(\Omega_T)} \leq l \|\Delta \psi\|^2_{L^2(\Omega_T)} \), we have

$$\sigma \|\Delta \psi\|_{L^2(\Omega_T)} \leq \varepsilon_1 \gamma \|\Delta F\|^2_{L^2(\Omega_T)}, \quad \sigma = \frac{k_s}{\varepsilon_1} > 0.$$  \hspace{1cm} (26)

Taking into account the Poincaré inequality, we arrive at the proof.

Theorem 3.1 Let the conditions of Theorem 2.1 hold. Then, for any initial data \( F(0) \in \mathcal{F} \) the sequence of iterations \( \{F^{(n)}\} \subset \mathcal{F} \), given by (23), converges to a quasisolution \( F^* \in \mathcal{F} \) of ISP1 in the norm of \( H^1(\Omega_T) \). The sequence of functionals \( \{J(F^{(n)})\} \) is a monotone decreasing and convergent one. Moreover, for the rate of convergence the following estimate holds

$$0 \leq J_1(F^{(0)}) - J_1(F^*) \leq 2L_1 d^{-1}, \quad d > 0, \quad n = 0, 1, 2, \ldots,$$

where \( L_1 > 0 \) is the Lipschitz constant defined in Lemma 3.1.

4. Adjoint problem approach for ISP2

We will define the weak solution of the direct problem (3) in the Sobolev space \( H^{1,1}(\Omega_T) \) as follows [22]: find such a function to be as the function \( u \in H^{1,1}(\Omega_T) \), with \( u(x, 0) = u_0(x) \), which satisfies the integral identity:

$$\int_\Omega \int_0^T (u_t \eta + ku_x \eta_x) \, dx \, dt = \int_\Omega \int_0^T F(x, t) \eta(x, t) \, dx \, dt, \quad \forall \eta \in H^{1,0}(\Omega_T).$$  \hspace{1cm} (27)

Here \( H^{1,0}(\Omega_T) := \{ \psi \in H^{1,0}(\Omega_T) : \psi(0, 0) = 0, \forall t \in (0, T) \} \).

Now let us assume \( F, F + \Delta F \in \mathcal{F} \), and calculate the first variation \( \Delta J_2(F) := J_2(F + \Delta F) - J_2(F) \) of the cost functional \( J_2(F) \). We have

$$\Delta J_2(F) = -2 \int_0^T [k(0)u_x(0, t; F) + f(t)] \Delta f(t) \, dt + \int_0^T [k(0)u_x(0, t; F)]^2 \, dt,$$  \hspace{1cm} (28)

where \( \Delta u(x, t; F) = u(x, t; F + \Delta F) - u(x, t; F) \in H^{1,1}(\Omega_T) \), \( \Delta f(t) := -[k(0)u_x(0, t; F + \Delta F) - k(0)u_x(0, t; F)] \) or \( \Delta u := \Delta u(x, t; F) \) is the solution of the following parabolic problem:

$$\begin{cases}
\Delta u_t = (k(x) \Delta u_x)_x + \Delta F(x, t), & (x, t) \in \Omega_T, \\
\Delta u(x, 0) = 0, & x \in (0, l), \\
\Delta u(0, t) = 0, \quad \Delta u_x(t, l) = 0, & t \in (0, T].
\end{cases}$$  \hspace{1cm} (29)

Lemma 4.1 Let \( u(x, t; F), u(x, t; F + \Delta F) \in H^{1,1}(\Omega_T) \) be solutions of the direct problem (3) for the given sources \( F, F + \Delta F \in \mathcal{F} \). Suppose that \( f(t) := -k(0)u_x(0, t; F) \),
then for all \( \tau \in (0, T) \) the following integral identity holds:

\[
\int_\Omega \int_0^\tau \phi(x, t) \Delta F(x, t) \, dx \, dt = -\int_0^\tau \Delta f(t) q(t) \, dt, \quad \forall q(t) \in H^0[0, T].
\]  

**Proof** Multiplying both the sides of Equation (29) by an arbitrary function \( \psi(x, t) \) and applying by parts integration formula, we obtain

\[
- \int_\Omega \int_0^\tau \left[ \phi(x, t; F) + (k(x) \phi_x(x, t; F))_x \right] \Delta u(x, t; F) \, dx \, dt \\
+ \int_0^\tau \left[ \Delta u(x, t; F) \phi(x, t) \right] x=0 \, dx - \int_0^\tau \left[ k(x) \Delta u_x(x, t; F) \phi(x, t) - k(x) \Delta u(x, t; F) \phi_x(x, t) \right] x=0 \, dt
= \int_\Omega \int_0^\tau \Delta F(x, t) \phi(x, t) \, dx \, dt
\]

Assuming that the function \( \phi(x, t) \) is the solution of problem (30), and then taking into account to the initial/final and boundary conditions in (29) and (30), we obtain identity (31).

**Remark 4.1** The well-posed backward parabolic problem (30) is defined to be the adjoint problem, corresponding to ISP2.

Let us now choose the arbitrary control function \( q(t) \in H^0[0, T] \) in (30) as follows: \( q(t) = 2[k(0)u_x(0, t; F) + f(t)] \), where \( f(t) \) is the measured output data given by (4). Substituting this in the integral identity (31), we get

\[
\int_\Omega \int_0^\tau \Delta F(x, t) \phi(x, t) \, dx \, dt = -2 \int_0^T [k(0)u_x(0, t; F) + f(t)] \Delta f(t) \, dt.
\]

This, with the first variation formula (28), implies

\[
\Delta J_2(F) = \int_\Omega \int_0^\tau \phi(x, t) \Delta F(x, t) \, dx \, dt + \int_0^T [k(0)u_x(0, t; F)]^2 \, dt.
\]  

To prove that the function \( \psi(x, t) \) is the of Fréchet differential of the cost functional \( J_2(F) \), we need to show \( \|k(0) \Delta u_x(0, \cdot; F) \|_{IP(0, T)} = o(\|\Delta F\|_{IP(\Omega_T)}) \). For this aim we need some preliminary estimates for the solution \( \Delta u(x, t; F) \in H^{1,1}(\Omega_T) \) of the parabolic problem (29).

**Lemma 4.2** Let the conditions given in (8) hold. Then for the solution \( \Delta u(x, t; F) \in H^{1,1}(\Omega_T) \) of the parabolic problem (29) the following estimate holds:

\[
\|k(0) \Delta u_x(0, \cdot; F) \|^2_{IP[0, T]} \, dx \leq 2\left\{ \|\Delta u_t\|^2_{IP[\Omega_T]} + \|\Delta F\|^2_{IP[\Omega_T]} \right\}, \quad l > 0.
\]  

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\( f(t) + \Delta f(t) = -k(0)u_x(0, t; F + \Delta F) \) are the corresponding outputs. If \( \phi(x, t; F) \in H^{1,1}(\Omega_T) \) is the solution of the backward parabolic problem

\[
\begin{aligned}
\phi_t &= -(k(x)\phi_x)_x, \\
\phi(x, \tau) &= 0, \\
\phi(0, t) &= q(t), \\
\phi_x(l, t) &= 0, \\
\end{aligned}
\]  

then for all \( \tau \in (0, T) \) the following integral identity holds:

\[
\int_\Omega \int_0^\tau \phi(x, t) \Delta F(x, t) \, dx \, dt = -\int_0^\tau \Delta f(t) q(t) \, dt, \quad \forall q(t) \in H^0[0, T].
\]
Proof Integrating both the sides of Equation (29) on \([0, t]\) and using the boundary conditions, we get
\[
\int_0^t \Delta u_t(x, t) \, dx = -k(0) \Delta u_x(0, t; F) + \int_0^t \Delta F(x, t) \, dx, \quad \forall t \in (0, T).
\]

With the Cauchy–Bunyakovsky–Schwarz inequality, this yields
\[
[k(0) \Delta u_x(0, t; F)]^2 = \left\{ -\int_0^t \Delta u_t(x, t) \, dx + \int_0^t \Delta F(x, t) \, dx \right\}^2 \\
\leq 2\left\{ \left( \int_0^t \Delta u_t(x, t) \, dx \right)^2 + \left( \int_0^t \Delta F(x, t) \, dx \right)^2 \right\} \\
\leq 2\left\{ \int_0^t [\Delta u_t(x, t)]^2 \, dx + \int_0^t [\Delta F(x, t)]^2 \, dx \right\}
\]
for all \(t \in (0, T)\). Integrating both the sides on \([0, T]\), we obtain the proof.

\[\blacksquare\]

**Lemma 4.3** If the conditions given in (8) hold, then for the solution \(\Delta u(x, t; F) \in H^{1,1}(\Omega_T)\) of the parabolic problem (29) the following estimate holds:
\[
||\Delta u_t||_{H^2(\Omega_T)}^2 \leq \frac{\varepsilon_2^2}{2\varepsilon_2 - 1} ||\Delta F||_{H^2(\Omega_T)}^2, \quad \forall \varepsilon_2 > 1/2.
\]  

**Proof** Since \(\eta(x, t)\) in (27) is in \(H^{1,0}(\Omega_T)\) and the weak solution \(\Delta u(x, t)\) of the parabolic problem (29) is defined in \(H^{1,1}(\Omega_T) \subset H^{1,0}(\Omega_T)\), we may assume \(\eta = \Delta u_t \in H^{1,0}(\Omega_T)\) in (27). Then multiplying both the sides of the parabolic equation (29) by \(\eta = \Delta u_t\) and taking into account the conditions \(\Delta u_x(l, t) = 0, \Delta u_t(0, t) = 0\), we obtain
\[
\int \int_{\Omega_T} [\Delta u_t]^2 \, dx \, dt + \int \int_{\Omega_T} k(x) \Delta u_x \Delta u_{xx} \, dx \, dt = \int \int_{\Omega_T} \Delta F(x, t) \Delta u_t(x, t) \, dx \, dt.
\]  

Let us use the following identity in the second left-hand side integral
\[
\int_0^T \Delta u_x(x, t) \Delta u_{xt}(x, t) \, dt = \frac{1}{2} \int_0^T \left[\Delta u_x(x, t)\right]^2 \, dt = \frac{1}{2} \left[\Delta u_x(x, T)\right]^2, \quad \forall x \in (0, l).
\]

Using this in (35), we conclude that
\[
\int \int_{\Omega_T} [\Delta u_t(x, t)]^2 \, dx \, dt + \frac{1}{2} \int_0^l k(x) [\Delta u_x(x, T)]^2 \, dx = \int \int_{\Omega_T} \Delta F(x, t) \Delta u_t(x, t) \, dx \, dt.
\]

Hence
\[
\int \int_{\Omega_T} [\Delta u_t(x, t)]^2 \, dx \, dt \leq \int \int_{\Omega_T} \Delta F(x, t) \Delta u_t(x, t) \, dx \, dt.
\]

Applying the Cauchy \(\varepsilon\)-inequality to the right-hand side, we obtain the proof.  

\[\blacksquare\]

The above two lemmas imply the following corollary.
Corollary 4.1 Let the conditions given in (8) hold, and \( \Delta u(x, t; F) \in H^{1,1}(\Omega_T) \) is the solution of the parabolic problem (29). Then the following estimate holds:

\[
\|k(0)\Delta u_x(0, \cdot; F)\|^2_{L^2([0, T])} \leq \gamma_2 \|\Delta F\|^2_{L^2(\Omega_T)}, \quad \gamma_2 = 2[(\varepsilon_2^2/(2\varepsilon_2 - 1) + 1], \quad \varepsilon_2 > 1/2.
\]

This corollary, with (32), shows that the cost functional \( J_2(F) \) corresponding to ISP2 is Fréchet differentiable.

Theorem 4.1 Let the conditions given in (8) hold and \( \mathcal{F} \subset H^1(\Omega_T) \). Then the cost functional \( J_2(F) \) defined by (7) is Fréchet-differentiable, \( J_2(F) \in C^1(\mathcal{F}) \). Moreover, Fréchet derivative at \( F \in \mathcal{F} \) of the cost functional \( J_2(F) \) is defined via the solution \( \phi \in H^{1,1}(\Omega_T) \) of the adjoint problem

\[
\begin{align*}
\phi_t & = -(k(x)\phi_x)_x, & (x, t) \in \Omega_T, \\
\phi(x, T) & = 0, & x \in (0, l), \\
\phi(0, t) & = 2[k(0)u_x(0, t; F) + f(t)], & \phi_x(l, t) = 0, & t \in (0, T],
\end{align*}
\]

as follows:

\[
J'_2(F) = \phi(x, t; F).
\]

Having the estimate \( \|\Delta\phi(\cdot, \cdot; F)\|_{L^2(\Omega_T)} \leq 2C_0\|k(0)\Delta u_x(0, \cdot; F)\|_{L^2([0, T])}, \quad C_0 > 0 \), for the solution \( \Delta\phi(x, t) \in H^{1,1}(\Omega_T) \) of the problem

\[
\begin{align*}
\Delta\phi_t & = -(k(x)\Delta\phi_x)_x, & (x, t) \in \Omega_T, \\
\Delta\phi(x, T) & = 0, & x \in (0, l), \\
\Delta\phi(0, t) & = 2[k(0)\Delta u_x(0, t; F)], & \Delta\phi_x(l, t) = 0, & t \in (0, T],
\end{align*}
\]

and using then Corollary 4.1, one can prove the result similar to Lemma 3.1. Here \( \Delta\phi(x, t; F) := \phi(x, t; F + \Delta F) - \phi(x, t; F) \in H^{1,1}(\Omega_T) \).

5. Adjoint problem approach for ISP3

Consider the inverse source problem (5)–(6) with final time overdetermination. We will define the weak solution of the direct problem (5) as in (9), \( V^{1,0}(\Omega_T) := \{v \in V^{1,0}(\Omega_T) : \nu(0, t) = 0, \quad \forall t \in (0, T]\} \). To derive the gradient formula for the cost functional \( J_2(F) \), defined by (7), and prove its Lipschitz continuity, consider the first variation of this cost functional. We have

\[
\Delta J_2(F) = 2 \int_0^l [u(x, T; F) - u_T(x)] \Delta u(x, T; F) \, dx + \int_0^l [\Delta u(x, T; F)]^2 \, dx,
\]

where \( \Delta u(x, t; F) = u(x, t; F + \Delta F) - u(x, t; F) \in V^{1,0}(\Omega_T) \). Then the function \( \Delta u := \Delta u(x, t; F) \) is the solution of the following parabolic problem:

\[
\begin{align*}
\Delta u_t & = (k(x)\Delta u_x)_x + \Delta F(x, t), & (x, t) \in \Omega_T, \\
\Delta u(x, 0) & = 0, & x \in (0, l), \\
\Delta u(0, t) & = 0, & \Delta u_x(l, t) = 0, & t \in (0, T].
\end{align*}
\]
Proof Let us use the final time condition given in problem (43) to transform the right-hand side of (42):

\[
\int \int_{\Omega_T} \varphi(x, t; F) \Delta F(x, t) dx \, dt = 2 \int_0^1 [u(x, T; F) - u_T(x)] \Delta u(x, T; F) dx, \quad \forall F \in \mathcal{F}, \quad (42)
\]

where \( \varphi(x, t; F) \) is the solution of the following backward parabolic problem:

\[
\begin{aligned}
\varphi_t &= -(k(x)\varphi_x)_x, \quad (x, t) \in \Omega_T, \\
\varphi(x, T) &= 2[u(x, T; F) - u_T(x)], \quad x \in (0, l), \\
\varphi(0, t) &= 0, \quad \varphi_x(l, t) = 0, \quad t \in (0, T].
\end{aligned}
\]

\[ \text{Proof} \]

Let us use the final time condition given in problem (43) to transform the right-hand side of (42):

\[
2 \int_0^1 [u(x, T; F) - \mu(x)] \Delta u(x, T; F) dx
\]

\[
= \int_0^1 \varphi(x, T; F) \Delta u(x, T; F) dx
\]

\[
= \int_0^1 \left\{ \int_0^T (\varphi(x, t; F) \Delta u(x, t; F))_t dt \right\} dx
\]

\[
= \int_0^T \int_{\Omega_T} \left\{ \varphi(x, t; F) \Delta u(x, t; F) + \varphi(x, t; F) \Delta u_t(x, t; F) \right\} dx \, dt
\]

\[
+ \int_0^T \int_{\Omega_T} \varphi(x, t; F) \Delta F(x, t) dx \, dt
\]

\[
= \int_0^T \left\{ -k(x)\varphi_x(x, t; F) \Delta u(x, t; F) + \varphi(x, t; F)k(x)\Delta u_x(x, t; F) \right\}_{x=0}^{x=l} dt
\]

\[
+ \int_0^T \int_{\Omega_T} \varphi(x, t; F) \Delta F(x, t) dx \, dt.
\]

Taking into account the boundary conditions given in (41) and (43), we obtain the proof. \[ \square \]

Remark 5.1 The backward parabolic problem (43) is defined to be as the adjoint problem, corresponding to ISP3.

Now we use the integral identity (42) on the right-hand side of formula (40) for the first variation of the cost functional \( J_3(F) \). Then we have

\[
\Delta J_3(F) = \int \int_{\Omega_T} \varphi(x, t; F) \Delta F(x, t) dx \, dt + \int_0^1 [\Delta u(x, T; F)]^2 dx. \quad (44)
\]

To estimate the last term, we need the following lemma.
**Lemma 5.2** For the solution \( \Delta u \in V^{1,0}(\Omega_T) \) of the parabolic problem (41), the following estimate holds:

\[
\int_{0}^{l} [\Delta u(x, T; F)]^2 \, dx \leq \frac{1}{\varepsilon_3} \| \Delta F \|^2_{H^1(\Omega_T)}, \quad \forall \Delta F \in \mathcal{F},
\]

where \( \varepsilon_3 \in (0, 4k_*/f^2) \), \( k_* = \min_{[0,l]} > 0 \).

**Proof** Multiplying both the sides of the parabolic equation (41) by \( \Delta u \), integrating on \( \Omega_T \) and using the initial and boundary conditions, we obtain the following energy identity:

\[
\int_{\Omega_T} k(x) (\Delta u(x, t; F))^2 \, dx \, dt + \frac{1}{2} \int_{0}^{l} [\Delta u(x, T; F)]^2 \, dx = \int_{\Omega_T} \Delta F(x, t) \Delta u(x, t; F) \, dx \, dt.
\]

Applying the Cauchy \( \varepsilon \)-inequality to the right-hand side integral and then using the Poincaré inequality, we obtain

\[
(k_* - \varepsilon_3 f^2/4) \int_{\Omega_T} [\Delta u(x, t; F)]^2 \, dx \, dt + \frac{1}{2} \int_{0}^{l} [\Delta u(x, T; F)]^2 \, dx \leq \frac{1}{\varepsilon_3} \int_{\Omega_T} [\Delta F(x, t)]^2 \, dx \, dt.
\]

Choosing the arbitrary parameter \( \varepsilon_3 > 0 \) as \( \varepsilon < 4k_* f^2/ \), we conclude that \( k_* - \varepsilon_3 f^2/4 > 0 \).

The positivity of the term \( k_* - \varepsilon_3 f^2/4 \) implies the proof. \( \blacksquare \)

The lemma implies that the last integral in (44) is bounded by the term \( o(\| \Delta F \|^2_{H^1(\Omega_T)}) \).

**Theorem 5.1** Let the conditions given in (8) hold and \( \mathcal{F} \subset H^1(\Omega_T) \). Then the cost functional \( J_3(F) \) defined by (7) is Fréchet-differentiable, \( J_3(F) \in C^1(\mathcal{F}) \). Moreover, Fréchet derivative at \( F \in \mathcal{F} \) of the cost functional \( J_3(F) \) is defined via the solution \( \varphi \in V^{1,0}(\Omega_T) \) of the adjoint problem (43) as follows:

\[
J_3'(F) = \varphi(x, t; F), \quad (45)
\]

**Corollary 5.1** Let \( J_3(F) \in C^1(\mathcal{F}) \) and \( \mathcal{F}_* \subset \mathcal{F} \) be the set of quasisolutions of ISP3. Then, \( F_\star \in \mathcal{F}_* \) is a strict solution of ISP3 if and only if \( \psi(x, t; F_\star) \equiv 0, \) on \( \Omega_T \).

The following result shows that the gradient of the cost functional \( J_3(F) \) is Lipschitz continuous.

**Lemma 5.3** Let the conditions of (8) hold. Then the functional \( J_3(F) \) is of Hölder class \( C^{1,1}(\mathcal{F}) \) and

\[
\| J_3'(F + \Delta F) - J_3'(F) \|_{H^1(\Omega_T)} \leq L_3 \| \Delta F \|_{H^1(\Omega_T)}, \quad \forall F, F + \Delta F \in \mathcal{F}, \quad (46)
\]

where

\[
\| J_3'(F + \Delta F) - J_3'(F) \|_{H^1(\Omega_T)} := \| \Delta \varphi(x, t; F) \|_{H^1(\Omega_T)},
\]

and the Lipschitz constant \( L_3 > 0 \) is defined as follows:

\[
L_3 = 1/\sqrt{2k_1 \varepsilon_3}, \quad \varepsilon_3 \in (0, 4k_* f^2).
\]

**Proof** The function \( \Delta \varphi(x, t; F) := \varphi(x, t; F + \Delta F) - \varphi(x, t; F) \in V^{1,0}(\Omega_T) \) is the solution of the following problem:

\[
\begin{align*}
\Delta \varphi_t &= -(k(x)\Delta \varphi_x)_x, & (x, t) \in \Omega_T, \\
\Delta \varphi(x, T) &= 2\Delta u(x, T; F), & x \in (0, l), \\
\Delta \varphi(0, t) &= 0, \quad \Delta \varphi_x(l, t) = 0, & t \in (0, T).
\end{align*}
\]
Multiplying both the sides of the above equation by \( \Delta \phi(x, t; F) \), integrating on \( \Omega_T \) and using the initial and boundary conditions, we obtain the following energy identity:

\[
\int \int_{\Omega_T} k(x)(\Delta \phi(x, t; F))^2 \, dx \, dt + \frac{1}{2} \int_0^t (\Delta \phi(x, 0; F))^2 \, dx = 2 \int_0^t [\Delta u(x, T; F)]^2 \, dx.
\]

This identity implies

\[
k_0 \int \int_{\Omega_T} (\Delta \phi(x, t; F))^2 \, dx \, dt \leq 2 \int_0^t (\Delta u(x, T; F))^2 \, dx.
\]

Applying the Poincaré inequality to the left-hand side and using Lemma 5.2, we obtain the proof. \( \Box \)

Having Fréchet differentiability of the cost functional \( J_3(F) \) and Lipschitz continuity of its gradient, we can now apply Theorem 3.1 also for ISP3.

**THEOREM 5.2** Let the conditions of (8) hold. Then, for any initial data \( F^{(0)} \in \mathcal{F} \) the sequence of iterations \( \{F^{(n)}\} \subset \mathcal{F} \), given by the iteration process

\[
F^{(n+1)} = F^{(n)} - \alpha_n J_3(F^{(n)}) \quad n = 0, 1, 2, \ldots,
\]

converges in the norm of \( H^1(\Omega_T) \) to a quasisolution \( F^* \in \mathcal{F}^* \) of ISP3. Moreover, the sequence of functionals \( \{J_3(F^{(n)})\} \) is a monotone decreasing and convergent one, and for the rate of convergence the following estimate holds:

\[
0 \leq J_3(F^{(n)}) - J_3(F_*) \leq 2L_3d^{-1}n^{-1}, \quad d > 0, \quad n = 0, 1, 2, \ldots,
\]

where \( L_3 > 0 \) is the Lipschitz constant defined in Lemma 5.3.

6. Convexity of cost functionals and unicity of solutions of the inverse source problems

Having via the solutions of appropriate adjoint problems the explicit formulas (22), (38) and (45) for the gradients of the cost functionals of the considered inverse source problem, one can construct a relationship between the minimization problems for these functionals and the corresponding variational inequalities. Then the well-known results of convex analysis can be used [26]. Indeed, if \( \mathcal{F} \) is a closed convex set and \( J(F) \in C^1(\mathcal{F}) \), then the necessary and sufficient condition for \( F^* \in \mathcal{F}^* \) is the fulfilment of the the variational inequality:

\[
\langle J'(F_+), F - F_+ \rangle \geq 0, \quad \forall F \in \mathcal{F},
\]

where \( \mathcal{F}^* \subset \mathcal{F} \) is the set of solutions of the minimization problem \( J(F^*) = \min_{F \in \mathcal{F}} J(F) \). On the other hand, if the functional \( J(F) \in C^1(\mathcal{F}) \) is strictly convex, then the operator \( J': \mathcal{F} \mapsto H^1 \) is strictly monotone, i.e.

\[
\langle J'(F + \Delta F) - J'(F), \Delta F \rangle > 0, \quad \forall F, F + \Delta F \in \mathcal{F}.
\]

In this case the minimization problem has a unique solution.

Let us analyse the above-considered inverse source problems in the sense of fulfilment of the strictly monotonicity condition (48).

First, consider the cost functional \( J_1(F) \) corresponding to ISP1. To establish the strict convexity of the cost functional \( J_1(F) \in C^{1,1}(\mathcal{F}) \), we need the following lemma.
Lemma 6.1 Let the conditions of (8) hold. Assume that $F,F+\Delta F \in \mathcal{F}$. Then for the cost functional $J_1(F) \in C^{1,1}(\mathcal{F})$ the following formula is valid:

$$
\langle J'_1(F + \Delta F) - J'_1(F), \Delta F \rangle_{H^1(\Omega)} = 2 \int_0^T [\Delta u(0, t; F)]^2 \, dt, \quad \forall F,F + \Delta F \in \mathcal{F},
$$

where $\Delta u(0, t; F) := \Delta u(x, t; F)|_{x=0}$, and $\Delta u(x, t; F) \in \dot{V}^{1,0} (\Omega_T)$ is the solution of the parabolic problem (12).

Proof Using the gradient formula (22) on the left-hand side of (48) and then taking into account that $\Delta u(x, t)$ is the solution of the parabolic problem (12), we obtain

$$
\langle J'_1(F + \Delta F) - J'_1(F), \Delta F \rangle_{H^1(\Omega)} = \int \int_{\Omega_T} \Delta \psi(x, t; F) \Delta F(x, t) \, dx \, dt
$$

$$
= \int \int_{\Omega_T} \Delta u_t \Delta \psi \, dx \, dt - \int \int_{\Omega_T} (k(x) \Delta u_x)_x \Delta \psi \, dx \, dt,
$$

where $\Delta \psi(x, t; F) \in \dot{V}^{1,0} (\Omega_T)$ is the solution of the backward parabolic problem (25). Applying integration by parts formula to the right-hand side integrals, using the initial and boundary conditions in (12) and (25), we obtain the desired result (49).

Now consider the cost functional $J_2(F)$ corresponding to ISP2.

Lemma 6.2 Let the conditions of (8) hold. Assume that $F,F+\Delta F \in \mathcal{F}$. Then for the cost functional $J_2(F) \in C^{1,1}(\mathcal{F})$ the following formula is valid:

$$
\langle J'_2(F + \Delta F) - J'_2(F), \Delta F \rangle_{H^1(\Omega)} = 2k^2(0) \int_0^T [\Delta u_3(0, t; F)]^2 \, dt, \quad \forall F,F + \Delta F \in \mathcal{F},
$$

where $\Delta u_3(0, t; F) := \Delta u(x, t; F)|_{x=0}$, and $\Delta u(x, t; F) \in H^{1,1} (\Omega_T)$ is the solution of the parabolic problem (29).

Proof Using the gradient formula (38) on the left-hand side of (49), and then taking into account that $\Delta u(x, t)$ is the solution of the parabolic problem (29), we obtain

$$
\langle J'_2(F + \Delta F) - J'_2(F), \Delta F \rangle_{H^1(\Omega)} = \int \int_{\Omega_T} \Delta \phi(x, t; F) \Delta F(x, t) \, dx \, dt
$$

$$
= \int \int_{\Omega_T} \Delta u_t \Delta \phi \, dx \, dt - \int \int_{\Omega_T} (k(x) \Delta u_x)_x \Delta \phi \, dx \, dt,
$$

where $\Delta \phi(x, t; F) \in H^{1,1} (\Omega_T)$ is the solution of the backward parabolic problem (39). Applying integration by parts formula to the right-hand side integrals, using the initial and boundary conditions in (29) and (39), we obtain the proof.

Finally, consider the cost functional $J_3(F)$ corresponding to ISP3 with final data overdetermination.

Lemma 6.3 Let the conditions of (8) hold. Assume that $F,F+\Delta F \in \mathcal{F}$. Then for the cost functional $J_3(F) \in C^{1,1}(\mathcal{F})$ the following formula is valid:

$$
\langle J'_3(F + \Delta F) - J'_3(F), \Delta F \rangle_{H^1(\Omega)} = 2 \int_0^T [\Delta u(x, T; F)]^2 \, dt, \quad \forall F,F + \Delta F \in \mathcal{F},
$$

where $\Delta u(x, t; F) \in H^{1,1} (\Omega_T)$ is the solution of the parabolic problem (39). Applying integration by parts formula to the right-hand side integrals, using the initial and boundary conditions in (29) and (39), we obtain the proof.
where $\Delta u(x, T; F) := \Delta u(x, t; F)|_{t=T}$, and $\Delta u(x, t; F) \in V^{1,0}(\Omega_T)$ is the solution of the parabolic problem (41).

**Proof** Using the gradient formula (45) on the left-hand side of (50), and then taking into account that $\Delta u(x, t)$ is the solution of the parabolic problem (41), we obtain

$$
\langle J_3(F + \Delta F) - J_3(F), \Delta F \rangle_{H^p(\Omega)} = \int \int_{\Omega_T} \Delta \varphi(x, t; F) \Delta F(x, t) \, dx \, dt
$$

$$
= \int \int_{\Omega_T} \Delta u, \Delta \varphi \, dx \, dt - \int \int_{\Omega_T} (k(x)\Delta u_x), \Delta \varphi \, dx \, dt,
$$

where $\Delta \varphi(x, t; F) \in V^{1,0}(\Omega_T)$ is the solution of the backward parabolic problem (47). Applying integration by parts formula to the right-hand side integrals, using the initial and boundary conditions in (41) and (47), we obtain formula (50).

The above three lemmas imply the necessary and sufficient conditions the strict convexity of the cost functionals corresponding to the considered inverse source problems.

**Corollary 6.1** The cost functionals $J_1(F)$, $J_2(F)$, $J_3(F)$ are strictly convex if and only if the conditions

$$
\int_0^T [u(0, t; F_1) - u(0, t; F_2)]^2 \, dt > 0,
$$

$$
\int_0^T [u_x(0, t; F_1) - u_x(0, t; F_2)]^2 \, dt > 0,
$$

$$
\int_0^T [u(x, T; F_1) - u(x, T; F_2)]^2 \, dx > 0,
$$

$\forall F_1, F_2 \in \mathcal{F}$, where the functions $u(x, t; F_m)$ are the solutions of corresponding direct problems for the given source functions $F_m \in \mathcal{F}$, $m = 1, 2$.

Inequalities (52)–(54) in Corollary 6.1 can also be used as criteria for unicity as well as for degrees of ill-conditionedness of the considered inverse source problems.

**7. An inverse source problem for abstract parabolic equation with final data observation**

The above strict convexity conditions obtained by the adjoint problem approach can be completed in the strong form by using semigroup theory for abstract parabolic equations. Specifically, the representation obtained here for the input–output map corresponding to ISP3, allows to prove a sufficient condition for unicity and derive non-uniqueness structure of a solution.

Let $V$ be a separable Hilbert space. For each $t \in (0, T]$ we define the positive defined symmetric continuous bilinear functional $a(t; u, v) := (A(t)u, v)$ on $V$, generated by the linear self-adjoint operator $A(t) \in \mathcal{L}(V, V')$, where $V'$ is the dual of $V$. Further assume that $H$ is a Hilbert space which is identified with its dual $H'$, and the embedding $V \hookrightarrow H$ is dense and continuous. Hence $V \hookrightarrow H \hookrightarrow V'$. Consider the abstract Cauchy problem

$$
\begin{cases}
  u'(t) = Au + F(t), & t \in (0, T], \\
  u(0) = u_0,
\end{cases}
$$

(55)
where $F \in H^0(0, T; V')$ and $u_0 \in H$. Since the operator $A$ is assumed to be uniformly coercive, we will assume that there exists a positive number $\gamma_0 > 0$ such that

$$a(t; v, v) \geq \gamma_0 \|v\|^2, \quad \forall v \in V, \quad t \in (0, T].$$

According to general theory [3, p. 112], there exists a unique solution $u \in H^0(0, T; V)$ of the Cauchy problem (36), and it satisfies the following estimate:

$$\|v\|_{H^p(0, T; V)} \leq \frac{1}{\gamma_0} \left( \|F\|_{H^p(0, T; V')} + \|u_0\|_H \right).$$

Now let $F \in F \subset H^0(0, T; V')$ and $F$ be the set of admissible sources. The abstract inverse source problem, subsequently AISP consists of determining the unknown source term $F \in F$ in the Cauchy problem (55) from the final state observation (measured output data):

$$u_T := u(t)|_{t=T}, \quad T > 0. \quad (57)$$

In this context, the Cauchy problem (55) will be regarded as a direct problem.

To apply the semigroup method for AISP, we assume that $\{S(t): t \geq 0\}$ is a linear contraction semigroup consisting of the family of operators $\{S(t)\}, \ t \geq 0$, on $H$, whose infinitesimal generator is an extension of the operator $A$. Then for each given $F \in F$, the unique solution of the direct problem (55) can be represented as follows:

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau) F(\tau) d\tau, \quad t \in (0, T]. \quad (58)$$

We denote by $u(t; F)$ the unique solution of the direct problem (55), corresponding to the given source term $F \in F$. Then introducing the input–output map

$$\Phi : F \mapsto H, \quad \Phi F := u(t; F)|_{t=T}, \quad T > 0,$$

we can reformulate AISP as the following operator equation:

$$\Phi F = u_T, \quad F \in F, \quad u_T \in H.$$ 

Thus the considered inverse problem can be reduced to the problem of invertibility of the input–output map $\Phi$.

Let us substitute $t = T$ in the semigroup representation (58) and use the aditional condition (57). Then we have

$$\int_0^T S(T - \tau) F(\tau) d\tau = u_T - S(T) u_0, \quad T > 0. \quad (59)$$

A solution of the integral equation (59) will be defined as a mild solution (or simply, solution) of AISP. If $F \subset W^{1,1}([0, T]; V')$ then this solution will also be a solution of the abstract Cauchy problem (55) (see, [2,3]).

The principal tool in the analysis of solution of AISP is the following representation.
\textbf{Theorem 7.1} Let $u_0, u_T \in \mathcal{D}(A)$, and $g \in H^0(0, T; \mathcal{D}(A))$ be an arbitrary function. Then the function
\begin{equation}
\tilde{F}(t) = A(S(T) - I)^{-1}(u_T - S(T)u_0) - A(S(T) - I)^{-1} \int_0^T S(T - \tau)g(\tau)d\tau + g(t), \quad \forall t \in (0, T]
\end{equation}
is a solution of AISP.

Conversely, if $\tilde{F}(t) \in H^0(0, T; V')$ is any solution of AISP, then there exists such a function $g \in H^0(0, T; \mathcal{D}(A))$ that this solution can be represented by formula (60).

\textbf{Proof} Let us substitute the function $\tilde{F}(t)$, given by (60), on the left-hand side integral of (59) and calculate it. Then we have
\begin{equation}
\int_0^T S(T - \tau)\tilde{F}(\tau)d\tau = \int_0^T S(T - \tau)A(S(T) - I)^{-1}(u_T - S(T)u_0)d\tau
- \int_0^T S(T - \tau)A(S(T) - I)^{-1}d\tau
\cdot \int_0^T S(T - \tau)g(\tau)d\tau + \int_0^T S(T - \tau)g(\tau)d\tau.
\end{equation}

We use the identity [2]
\begin{equation}
\int_0^T S(\tau)v\,d\tau = A^{-1}(S(\tau) - I)v, \quad \forall v \in \mathcal{D}(A), \quad t \in (0, T]
\end{equation}
in the first and second right-hand side integrals; substituting here $t = T$, we have
\begin{equation}
\int_0^T S(T - \tau)\tilde{F}(\tau)d\tau = u_T - S(T)u_0 - I \cdot \int_0^T S(T - \tau)g(\tau)d\tau + \int_0^T S(T - \tau)g(\tau)d\tau.
\end{equation}

This shows that the function $\tilde{F}(t)$, given by (60), is the solution of the integral equation (59).

To prove the second part of the theorem, now assume that $F(t)$ is a solution of the integral equation (59). We introduce the function
\begin{equation}
g(t) = F(t) - A(S(T) - I)^{-1}(u_T - S(T)u_0) + v,
\end{equation}
where the arbitrary function $v \in \mathcal{D}(A)$ will be defined below. Acting by the semigroup operator $S(T - \tau)$ to both the sides of (62) and then integrating on $[0, T]$, we get
\begin{equation}
\int_0^T S(T - \tau)g(\tau)d\tau = \int_0^T S(T - \tau)\tilde{F}(\tau)d\tau - A(S(T) - I)^{-1}(u_T - S(T)u_0) \int_0^T S(T - \tau)d\tau
+ \int_0^T S(T - \tau)d\tau \cdot v.
\end{equation}

The first right-hand side integral here is $u_T - S(T)u_0$, according to (59). In the second and third right-hand side integrals we use the identity (61). Then we obtain
\begin{equation}
\int_0^T S(T - \tau)g(\tau)d\tau = u_T - S(T)u_0 - A(S(T) - I)^{-1}(u_T - S(T)u_0)A^{-1}(S(T) - I)
+ A^{-1}(S(T) - I)v.
\end{equation}
Hence the arbitrary element $v \in \mathcal{D}(A)$ is defined as follows:

$$v = A(S(T) - I)^{-1} \int_0^T S(T - \tau) g(\tau) d\tau.$$  

Substituting this in (62), we obtain the required representation (60). This completes the proof.

Now consider AISP with the time-independent source term $F(t) \equiv F_0, \forall t \in (0, T)$:

$$\begin{cases} 
  u'(t) = Au + F_0, & t \in (0, T), \\
  u(0) = u_0, & u(T) = u_T,
\end{cases}$$  

where $F_0 \in \mathcal{F}_0 \subset \mathcal{D}(A)$, and $\mathcal{F}_0$ is the set of time-independent source terms. Substituting $F(t) = F_0$ in (59), we get

$$\int_0^T S(T - \tau) F_0 d\tau = u_T - S(T)u_0, \quad T > 0.$$  

According to identity (61), this implies

$$A^{-1}(S(T) - I)F_0 = u_T - S(T)u_0, \quad T > 0.$$  

Hence, in this case we have a unique solution.

**Lemma 7.1** Let $u_0, u_T \in \mathcal{D}(A)$. Then the unique solution of the inverse problem (63) with the time-independent source term $F_0 \in \mathcal{D}(A)$ can be represented by the formula

$$F_0 = A(S(T) - I)^{-1}(u_T - S(T)u_0).$$  

Comparing this result with formula (60), we obtain the following corollary.

**Corollary 7.1** Let the conditions of Theorem 7.1 hold. Then any solution of AISP with time-dependent source term $\tilde{F}$ is the sum of two elements:

$$\tilde{F} = F_0 + F(t), \quad F_0 \in \mathcal{F}_0, \quad F \in \mathcal{F},$$  

where $F_0$ is the unique solution, given by (64), of the inverse source problem (63) with the time-independent source term $F_0 \in \mathcal{D}(A)$, and $F \in \mathcal{F}$, given by

$$F(t) = g(t) - A(S(T) - I)^{-1} \int_0^T S(T - \tau) g(\tau) d\tau,$$  

with an arbitrary function $g \in H^0(0, T; \mathcal{D}(A))$, is a solution of AISP

$$\begin{cases} 
  u'(t) = Au + F(t), & t \in (0, T), \\
  u(0) = 0, & u(T) = 0,
\end{cases}$$  

with the homogeneous initial and final data.

Formulas (64)–(66) show the structure of representation (60) of a solution of AISP.  

To illustrate unicity of the solution of ISP3, let us employ Fourier method to ISP3, given by (5)–(6), assuming here the non-zero initial data $u(x, 0) = u_0(x)$ and $k(x) \equiv$
Using separation of variables \(u(x, t) = X(x)T(t)\) in the direct problem (5), we obtain the following Fourier sine series:

\[
u(x, t) = \sum_{n=0}^{\infty} u_n(t) \sin \mu_n x,
\]

where \(\{\sin \mu_n x\}, \ n = 0, 1, \ldots\), are eigenfunctions corresponding to the eigenvalues \(\lambda_n = [(\pi/2 + \pi n)/l]^2\), \(\mu_n = \sqrt{\lambda_n}\) and

\[
u_n(t) = u_{0,n}e^{-\kappa_n t} + \int_0^t e^{-\kappa_n(t-\tau)}F_n(\tau)d\tau.
\]

The Fourier coefficients \(u_{0,n}\) and \(F_n(t)\) in (68) are defined as follows:

\[
u_{0,n} = \frac{2}{l} \int_0^l u_0(\xi) \sin \mu_n \xi \, d\xi, \quad F_n(t) = \frac{2}{l} \int_0^t F(\xi, t) \sin \mu_n \xi \, d\xi.
\]

Using the condition \(u_T(x) = u(x, T)\) in (67), we get

\[
\int_0^T e^{-\kappa_n(T-\tau)}F_n(\tau)d\tau = u_{T,n} - u_{0,n}e^{-\kappa_n T}, \quad n = 0, 1, 2, \ldots,
\]

where

\[
u_{T,n} = \frac{2}{l} \int_0^l u_T(\xi) \sin \mu_n \xi \, d\xi.
\]

Thus, ISP3 is transformed to the Fredholm integral equation of the first kind, which is ill-posed.

For some applications, the source terms for the transport equations might depend on the variable \(x \in (0, l)\): \(F(x, t) \equiv F(x)\). Then substituting \(F_n\) instead of \(F_n(t)\) in Equation (68), we obtain the following formula:

\[
F_n = \frac{k\lambda_n(u_{T,n} - u_{0,n}e^{-\kappa_n T})}{1 - e^{-\kappa_n T}}.
\]

Hence for the time-independent source function \(F(x)\) in ISP3, the Fourier coefficients \(F_n\) in the Fourier sine series

\[
F(x) = \sum_{n=0}^{\infty} F_n \sin \mu_n x,
\]

are defined uniquely by the explicit formula (69). Thus the unique solution of ISP3 is given by (69)–(70).

8. Conclusions
This article presents a systematic study aimed at understanding inverse source problems for heat conduction equation with most typical measured output data. First, we analyse the nature of these problems by applying three different approaches: adjoint problem approach, semigroup method and Fourier method. Second, we demonstrate bounds of applicability of these approaches/methods. Third, we discuss feasibility of the adjoint...
problem approach for all the considered inverse source problems by establishing an integral relationship between solutions of inverse problems and corresponding adjoint problems. For all problems we prove Fréchet differentiability of functionals and Lipschitz continuity of the gradients $J(F)$ of these functionals. We also propose unicity criteria for all the considered inverse problems via the output data. Semigroup analysis shows that for time-independent source term, ISP3 has a unique solution. In the constant heat conduction coefficient case, this solution then is derived in explicit form by using the Fourier method.

References


