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RELATIONS BETWEEN SUMMABILITY OF THE FOURIER COEFFICIENTS IN REGULAR SYSTEMS AND FUNCTIONS FROM SOME LORENTZ TYPE SPACES

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Abstract. Let Λ_β, $\beta > 0$, denote the Lorentz space equipped with the (quasi) norm

$$\|f\|_{\Lambda_\beta} := \left(\int\limits_0^1 \left(f^*(t) t \lambda\Big(\frac{1}{t}\Big) \right)^\beta \frac{dt}{t} \right)^{\frac{1}{\beta}}$$

for a function f on [0,1] and with λ positive and equipped with some additional growth properties. Some estimates of this quantity and some corresponding sums of Fourier coefficients are proved for the case with general orthonormal regular systems. Under certain circumstances even two sided estimates are obtained.

რეზიუმე. ვთქვათ Λ_β, $\beta > 0$, აღნიშნავს ლორენცის სივრცეს კვაზინორმით

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სადაც λ დადებითია და აკმაყოფილებს გარკვეულ დამატებით პირობებს. ნაშრომში მოცემულია ამ ნორმისა და გარკვეული ორთონორმირებული რეგულარული სისტემის მიმართ ფურიეს კოეფიციენტებისგან შედგენილი ჯამების შეფასებები. ზოგიერთ შემთხვევაში დადგენილია ორმხრივი შეფასებები.

1. Introduction

Let f be a measurable function on a measure space (Ω, μ), where μ is an additive positive measure.

The nonincreasing rearrangement f^* of a function f is defined as follows:

$$m(\sigma, f) := \mu \{x \in \Omega : |f(x)| > \sigma\},$$

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$$f^*(t) := \inf \{\sigma : m(\sigma, f) \leq t\} .$$

Let $0 < \beta \leq \infty$. Let the function f be integrable on $[0,1]$ and let λ be a nonnegative function on $[1,\infty]$.

The generalized Lorentz space Λ_β consists of the functions f on $[0,1]$ such that $\|f\|_{\Lambda_\beta} < \infty$, where

$$\|f\|_{\Lambda_\beta} := \left(\int_0^1 \left(f^*(t) t \lambda\Big(\frac{1}{t}\Big) \right)^\beta \frac{dt}{t} \right)^{\frac{1}{\beta}} \text{ for } 0 < \beta < \infty,$$

$$\|f\|_{\Lambda_\infty} := \sup_{0 \leq t \leq 1} f^*(t) t \lambda\Big(\frac{1}{t}\Big) \text{ for } \beta = \infty.$$

Let the function f be periodic with period 1 and let $\Phi = \{\varphi_n\}_{n=1}^\infty$ be an orthonormal system. The numbers

$$a_n = a_n(f) = \int_0^1 f(x)\overline{\varphi_n(x)}dx, \quad n \in N$$

are called the Fourier coefficients of the function f with respect to the system $\Phi = \{\varphi_n\}_{n=1}^\infty$. Some Hardy-Littlewood type inequalities were proved in the work [7] for the trigonometrical systems $\Phi = \{\varphi_n\}_{n=1}^\infty$:
If $1 < p < \infty$, then,

$$c_1 \sum_{k=1}^\infty k^{p-2} \left|\bar{a_k}\right|^p \leq \|f\|_{L_p[0,1]}^p \leq c_2 \sum_{k=1}^\infty k^{p-2} \left|k \Delta a_k\right|^p , \tag{1.1}$$

where $\bar{a_k} = \frac{1}{k} \left| \sum_{m=1}^k a_m \right|$ and $\Delta a_k = a_k - a_{k+1}$, $k \in N$. Further, these inequalities were proved for regular system in [8]. One main purpose of this paper is to derive inequalities analogous to those in (1.1) in the case of general regular systems $\Phi = \{\varphi_n\}_{n=1}^\infty$ and for generalized Lorentz spaces of type Λ_β.

Conventions. The letter c $(c_1, c_2, etc.)$ means a constant not dependent on the involved functions and it can be different in different occurences. Moreover, for $A, C > 0$ the notation $A \asymp C$ means that there exists positive constants a_1 and a_2 such that $a_1 A \leq C \leq a_2 A$.

The paper is organized as follows: In Section 2 we present and discuss our main results. The detailed proofs can be found in Section 3. Section 4 is reserved for some concluding remarks and examples.

2. Main Results

Let $\delta > 0$ and $\lambda(t)$ be a nonnegative function on $[1, \infty)$. We define the following classes:

$$B_\delta = \Big\{\lambda(t) : \lambda(t)t^{-\frac{1}{2}-\delta}\text{is an increasing function and}$$
$$\lambda(t)t^{-1+\delta}\text{is a decreasing function}\Big\},$$
$$D_\delta = \Big\{\lambda(t) : \lambda(t)t^{-\delta}\text{is an increasing function and}$$
$$\lambda(t)t^{-1+\delta}\text{is a decreasing function}\Big\}.$$

The classes B and D are defined as follows: $B = \cup_{\delta>0} B_\delta$, $D = \cup_{\delta>0} D_\delta$.

We say that the orthonormal system $\Phi = \{\varphi_k(x)\}_{k=1}^{\infty}$ is regular if there exists a constant B_0 such that

1) for every segment e from $[0, 1]$ and $k \in \mathbf{N}$ it yields that
$$\Big|\int_e \varphi_k(x)dx\Big| \le B_0 \min(|e|, 1/k),$$

2) for every segment w from $\mathbf{N}$ and $t \in (0, 1]$ we have that
$$\Big(\sum_{k\in w} \varphi_k(\cdot)\Big)^*(t) \le B_0 \min(|w|, 1/t),$$

where $\left(\sum_{k\in w} \varphi_k(\cdot)\right)^*(t)$ as usual denotes the nonincreasing rerrangement of function $\sum_{k\in w} \varphi_k(x)$. This concept was introduced and studied by E.D. Nursultanov [8].

Examples of regular systems are all trigonometrical systems, the Walsh sistem and Prise's system. Our next result concerning regular systems reads:

Theorem 2.1. *Let $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ be a orthonormal regular system and let $1 \le \beta \le \infty$.*
If $\lambda(t)$ belongs to the class D, then
$$\left(\sum_{n=1}^{\infty}\Big(\overline{a}_n\lambda(n)\Big)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} \le c\left(\int_0^1 \Big(f^*(t)t\lambda\Big(\frac{1}{t}\Big)\Big)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}, \tag{2.1}$$
where $\overline{a}_n = \sup_{r\ge n}\frac{1}{r}\left|\sum_{m=1}^{r} a_m(f)\right|$, and $a_m(f)$ are the Fourier coefficients with respect to the system Φ.

For the case $\lambda(t) = t^\gamma$ Theorem 2.1 implies a corresponding result in [7]. The inequality (2.1) for $\lambda(t)$ from the class B is reversed to the inequality in Theorem 1 (b) in [5]. In our next statement we shall prove the fact that in (2.1) the expression $\overline{a}_n$ on the left hand side cannot in general be replaced by the expression $\overline{|a|}_n = \frac{1}{n}\sum_{k=1}^{n}|a_k|$.

Proposition 2.1. *Let $\Phi = \{e^{2\pi ikx}\}_{k=1}^{\infty}$ and let $1 \le \beta \le \infty$. If $\lambda(t)$ belongs to the class B, then there exists a function f such that*

$$\left(\int_0^1 \left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^{\beta} \frac{dt}{t}\right)^{\frac{1}{\beta}} < \infty,$$

and

$$\left(\sum_{n=1}^{\infty} \left(\overline{|a_n|}\lambda(n)\right)^{\beta} \frac{1}{n}\right)^{\frac{1}{\beta}} = +\infty,$$

where $\overline{|a_n|} = \frac{1}{n}\sum_{k=1}^{n}|a_k|$. Here $a_k = a_k(f)$ are the Fourier coefficients of the function f for the trigonometrical system Φ.

A function ω in $\mathbb{R}^+$ is called regular (see [13]) if it satisfies

$$\frac{W(t)}{t} \le C\omega(t), \quad t > 0,$$

where $W(t) = \int_0^t \omega(\tau)d\tau$ and $C > 0$ independent of t.
Our next result reads:

Theorem 2.2. *Let $1 < \beta < \infty$ and $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ be a regular system. Let $f \sim \sum_{k=1}^{\infty} a_k\varphi_k$ and $\lambda^{-1}(t)$ belong to the class D. If $\lim_{n\to\infty}\lambda(n)a_n = 0$ and*

$$\left(\sum_{n=1}^{\infty} \left(|n\Delta a_n|\,\lambda(n)\right)^{\beta} \frac{1}{n}\right)^{\frac{1}{\beta}} < \infty,$$

then $f \in \Lambda_\beta$ and the following inequality

$$\|f\|_{\Lambda_\beta} \le c\left(\sum_{n=1}^{\infty} \left(|n\Delta a_n|\,\lambda(n)\right)^{\beta} \frac{1}{n}\right)^{\frac{1}{\beta}}$$

holds, where $\Delta a_n = a_n - a_{n+1}$, $n \in \mathbb{N}$.

We say that a sequence of complex numbers $\{a_k\}_{k=1}^{\infty}$ is generalized monotone if there exists some constant M such that, for any $k \in N$, it yields that

$$|a_n| \le M\frac{1}{n}\left|\sum_{k=1}^{n} a_k\right|. \tag{2.2}$$

Remark 2.1. *If the sequence $\{a_k\}$ is quasi- monotone, i.e. $a_k > 0$, $k \in N$ and there exists $m > 0$, such that $\left\{\frac{a_k}{k^m}\right\}$ is monotone nonincreasing, then it is generalized monotone.* In fact,

$$a_k \asymp \frac{1}{k}\left(\sum_{r=0}^{k} r^m\right)\frac{a_k}{k^m} \le \frac{1}{k}\sum_{r=0}^{k} r^m \frac{a_r}{r^m} = \frac{1}{k}\sum_{r=0}^{k} a_r.$$

The implication in the reversed direction does not in general hold as our next example shows.

Example 2.1. Let $k \in N$ and define

$$a_k = \begin{cases} 0, & \text{if} \quad k \text{ is even,} \\ \frac{1}{k}, & \text{if} \quad k \text{ is odd.} \end{cases}$$

This sequence is not quasi - monotone but obviously $a_k \leq 2\frac{1}{k}\sum a_m, \quad k \in N$, i.e. this sequence is generalized monotone.

We say that the sequence of complex numbers $a = \{a_k\}$ satisfies the condition P, if there exists some constant M, which does is not depend of k, such that for any $k \in N$ it yields that

$$a_k^* \leq M\overline{a_k}.$$

Remark 2.2. *If the sequence $\{a_k\}$ is generalized monotone, then it satisfies condition P.* In fact, when $0 \leq a_k \leq b_k, \ k \in N$, it obviously follows that $a_k^* \leq b_k^*, \ k \in N$. If $|a_k| \leq M\overline{a}_k = b_k, \quad k \in N$, i.e. $\{a_k\}$ is generalized monotone, then it follows that $a_k^* \leq b_k^*, \quad k \in N$, but $\{M\overline{a}_k\}$ is monotonically nonincreasing. Therefore $a_k^* \leq b_k^* = M\overline{a}_k, \quad k \in N$.

Finally, we state the following equivalence result for functions with Fourier coefficients satisfying the condition P.

Theorem 2.3. *Let $1 \leq \beta \leq \infty$, $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be a regular system and $\lambda(t)$ belong to the class B. If the Fourier coefficients of the function f on the system Φ satisfies the condition P, then*

$$\left(\sum_{n=1}^{\infty} (\overline{a}_n\lambda(n))^{\beta}\,\frac{1}{n}\right)^{\frac{1}{\beta}} \asymp \left(\int_0^1 \left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}.$$

3. Proofs

We present the proofs in the order so the corresponding result can be used in later proofs.

Proof the Theorem 2.1. Let $\lambda(t)$ be from the class D. This means that there exists $\delta > 0$ such that $\lambda(t)t^{-\delta}$ is an increasing function and $\lambda(t)t^{-1+\delta}$ is a decreasing function. Let the function f be such that

$$\left(\int_0^1 \left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}} < \infty.$$

Let $n \in N$, and note that

$$\overline{a}_n = \sup_{r\geq n}\frac{1}{r}\left|\sum_{m=1}^{r} a_m(f)\right| = \sup_{r\geq n}\frac{1}{r}\left|\sum_{m=1}^{r}\int_0^1 f(t)\overline{\varphi_m(t)}dt\right| =$$

$$= \sup_{r\geq n} \frac{1}{r}\Big|\int_0^1 f(t)\sum_{m=1}^{r}\overline{\varphi_m(t)}dt\Big| \leq \sup_{r\geq n}\frac{1}{r}\int_0^1 |f(t)|\,|D_r(t)|\,dt.$$

By using a well-known inequality concerning nonincreasing rerrangements we obtain that

$$\overline{a}_n \leq \sup_{r\geq n}\frac{1}{r}\int_0^1 f^*(t)D_r^*(t)dt.$$

Hence, by using the regularity condition that $D_r^*(t) \leq B\min(r, \frac{1}{t})$, we find that

$$\overline{a}_n \leq B\sup_{r\geq n}\int_0^1 f^*(t)\min(1,\frac{1}{tr})dt = B\int_0^1 f^*(t)\min\left(1,\frac{1}{tn}\right)dt.$$

Let $f(x) = f_0(x) + f_1(x)$, where

$$f_0(x) = \begin{cases} f(x) - f^*(\frac{1}{2n}), & \text{if} \quad |f(x)| > f^*\left(\frac{1}{2n}\right) \\ 0, & \text{if} \quad |f(x)| \leq f^*\left(\frac{1}{2n}\right), \end{cases}$$

$$f_1(x) = \begin{cases} f^*\left(\frac{1}{2n}\right), & \text{if} \quad |f(x)| \geq f^*\left(\frac{1}{2n}\right) \\ f(x), & \text{if} \quad |f(x)| < f^*\left(\frac{1}{2n}\right). \end{cases}$$

Here we use the following well-known inequality:

$$(f_0+f_1)^*(t) \leq f_0^*\left(\frac{t}{2}\right) + f_1^*\left(\frac{t}{2}\right).$$

Then

$$\overline{a}_n \leq \int_0^1 f_0^*\left(\frac{t}{2}\right)\min(1,\frac{1}{tn})dt + \int_0^1 f_1^*\left(\frac{t}{2}\right)\min(1,\frac{1}{tn})dt =$$

$$= 2\int_0^{\frac{1}{2}} f_0^*(s)\min(1,\frac{1}{2sn})ds + 2\int_0^{\frac{1}{2}} f_1^*(s)\min(1,\frac{1}{2sn})ds.$$

The first integral can be estimated as follows:

$$\int_0^{\frac{1}{2}} f_0^*(s)\min(1,\frac{1}{2sn})ds \leq \int_0^1 f_0^*(s)\min(1,\frac{1}{2sn})ds,$$

$$\int_0^{\frac{1}{2n}} f^*(s)ds - \frac{1}{2n}f^*\left(\frac{1}{2n}\right). \tag{3.1}$$

Similary, for the second integral we have that

$$\int_0^{\frac{1}{2}} f_1^*(s)\min(1,\frac{1}{2sn})ds \le \int_0^1 f_1^*(s)\min(1,\frac{1}{2sn})ds =$$

$$= \Bigg(\int_0^{\frac{1}{2n}} f^*\left(\frac{1}{2n}\right)ds + \int_{\frac{1}{2n}}^1 f^*(s)\frac{1}{2sn}ds\Bigg) =$$

$$= \frac{1}{2n}f^*\left(\frac{1}{2n}\right) + \frac{1}{2n}\int_{\frac{1}{2n}}^1 f^*(s)\frac{ds}{s}. \tag{3.2}$$

By combinig (3.1) and (3.2) we find that

$$\overline{a}_n \le 2\Bigg(\int_0^{\frac{1}{2n}} f^*(s)ds + \frac{1}{2n}\int_{\frac{1}{2n}}^1 f^*(s)\frac{ds}{s}\Bigg). \tag{3.3}$$

According to (3.3), we have that

$$J := \left(\sum_{n=1}^{\infty}\left(\overline{a}_n\lambda(n)\right)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} \le$$

$$\le 2\Bigg(\sum_{n=1}^{\infty}\Bigg(\lambda(n)\Bigg(\int_0^{\frac{1}{2n}} f^*(s)ds + \frac{1}{2n}\int_{\frac{1}{2n}}^1 f^*(s)\frac{ds}{s}\Bigg)\Bigg)^{\beta}\frac{1}{n}\Bigg)^{\frac{1}{\beta}},$$

which, by Minkowski's inequality, gives that

$$J \le 2c\Bigg(\Bigg(\sum_{n=1}^{\infty}\Bigg(\lambda(n)\int_0^{\frac{1}{2n}} f^*(s)ds\Bigg)^{\beta}\frac{1}{n}\Bigg)^{\frac{1}{\beta}} +$$

$$+\Bigg(\sum_{n=1}^{\infty}\Bigg(\frac{1}{2n}\lambda(n)\int_{\frac{1}{2n}}^1 f^*(s)\frac{ds}{s}\Bigg)^{\beta}\frac{1}{n}\Bigg)^{\frac{1}{\beta}}\Bigg) :=$$

$$:= c_1\left(I_1 + I_2\right).$$

First we consider I_1. Choose ε so that $-1-\frac{1}{\beta} < \varepsilon < \delta - 1 - \frac{1}{\beta}$. By using elementary estimates we find that

$$I_1 = \Bigg(\sum_{n=1}^{\infty}\Bigg(\lambda(n)\int_0^{\frac{1}{2n}} f^*(s)ds\Bigg)^{\beta}\frac{1}{n}\Bigg)^{\frac{1}{\beta}} = \Bigg(\sum_{n=1}^{\infty}\Bigg(\lambda(n)\int_{2n}^{\infty} f^*\left(\frac{1}{t}\right)\frac{dt}{t^2}\Bigg)^{\beta}\frac{1}{n}\Bigg)^{\frac{1}{\beta}} =$$

$$= \left(\sum_{n=1}^{\infty} \left(\lambda(n) \sum_{k=2n}^{\infty} \int_{k}^{k+1} f^* \left(\frac{1}{t} \right) \frac{dt}{t^2} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{n=1}^{\infty} \left(\lambda(n) \sum_{k=2n}^{\infty} f^* \left(\frac{1}{k+1} \right) \frac{1}{k^2} \int_{k}^{k+1} dt \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}}.$$

Here we estimate $\frac{1}{k^2}$ by $\frac{4}{(k+1)^2}$, apply Hölder's inequality and use that $-1 - \frac{1}{\beta} < \varepsilon < \delta - 1 - \frac{1}{\beta}$ to find that

$$I_1 \leq 4c \left(\sum_{n=1}^{\infty} \left(\lambda(n) \sum_{k=2n+1}^{\infty} f^* \left(\frac{1}{k} \right) \frac{1}{k^2} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \leq$$

$$\leq 4c \left(\sum_{n=1}^{\infty} \left(\lambda(n) \sum_{k=2n}^{\infty} f^* \left(\frac{1}{k} \right) \frac{1}{k^2} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \leq$$

$$\leq 4c \left(\sum_{n=1}^{\infty} \left(\lambda(n) \left(\sum_{k=2n}^{\infty} \left(f^* \left(\frac{1}{k} \right) k^{\varepsilon} \right)^{\beta} \right)^{\frac{1}{\beta}} \cdot \left(\sum_{k=2n}^{\infty} \left(\frac{1}{k^{\varepsilon+2}} \right)^{\beta'} \right)^{\frac{1}{\beta'}} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \asymp$$

$$\asymp \left(\sum_{n=1}^{\infty} \lambda^{\beta}(n) \left(\sum_{k=2n}^{\infty} \left(f^* \left(\frac{1}{k} \right) \cdot k^{\varepsilon} \right)^{\beta} \right)^{\frac{1}{\beta}} \cdot (2n)^{-(1+\varepsilon)\beta - 1} \frac{1}{n} \right)^{\frac{1}{\beta}} =$$

$$= \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) k^{\varepsilon} \right)^{\beta} \sum_{n=1}^{\frac{k}{2}} \frac{\lambda^{\beta}(n)}{n^{\delta\beta}} n^{\delta\beta} n^{-(1+\varepsilon)\beta - 2} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) k^{\varepsilon} \lambda(k) k^{-\delta} \right)^{\beta} \sum_{n=1}^{\frac{k}{2}} n^{(\delta - \varepsilon - 1)\beta - 2} \right)^{\frac{1}{\beta}} \asymp$$

$$\asymp \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) \frac{\lambda(k)}{k} \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}}.$$

Summing up we have proved that

$$I_1 \leq c \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) \frac{\lambda(k)}{k} \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}},$$

which similary as before implies that

$$I_1 \leq c \left(\int_{0}^{1} \left(f^*(t) \lambda \left(\frac{1}{t} \right) t \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}}. \tag{3.4}$$

Now we will derive a similar estimate for I_2. Choose $\varepsilon > 0$ such that $-\frac{1}{\beta} - \delta < \varepsilon < -\frac{1}{\beta}$, where $\delta > 0$. By using elementary estimates and Hölder's inequality, we see that

$$I_2 = \left(\sum_{n=1}^{\infty} \left(\lambda(n) \frac{1}{2n} \int_{\frac{1}{2n}}^{1} f^*(s) \frac{ds}{s} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} =$$

$$= \left(\sum_{n=1}^{\infty} \left(\lambda(n) \frac{1}{2n} \int_{1}^{2n} f^* \left(\frac{1}{t} \right) \frac{dt}{t} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} =$$

$$= \left(\sum_{n=1}^{\infty} \left(\lambda(n) \frac{1}{2n} \sum_{k=1}^{2n} \int_{k}^{k+1} f^* \left(\frac{1}{t} \right) \frac{dt}{t} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{n=1}^{\infty} \left(\lambda(n) \frac{1}{2n} \sum_{k=1}^{2n} f^* \left(\frac{1}{k+1} \right) \frac{1}{k} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{n=1}^{\infty} \left(\lambda(n) \frac{1}{n} \sum_{k=1}^{2n} f^* \left(\frac{1}{k+1} \right) \frac{1}{k+1} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{n=1}^{\infty} \left(\frac{\lambda(n)}{n} \left(\sum_{k=1}^{2n+1} \left(f^* \left(\frac{1}{k} \right) k^{\varepsilon} \right)^{\beta} \right)^{\frac{1}{\beta}} \left(\sum_{k=1}^{2n+1} \left(\frac{1}{k^{(1+\varepsilon)\beta'}} \right) \right)^{\frac{1}{\beta'}} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \asymp$$

$$\asymp \left(\sum_{n=1}^{\infty} \left(\frac{\lambda(n)}{n} \right)^{\beta} (2n+1)^{-\varepsilon\beta - 1} \sum_{k=1}^{2n+1} \left(f^* \left(\frac{1}{k} \right) k^{\varepsilon} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}}.$$

Hence, by interchanging the order of summation and using the assumptions, we find that

$$I_2 \leq c \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) k^{\varepsilon} \right)^{\beta} \sum_{n=\left[\frac{k-1}{2}\right]}^{\infty} \frac{\lambda^{\beta}(n)\, n^{-(1+\varepsilon)\beta - 2} n^{(-1+\delta)\beta}}{n^{(-1+\delta)\beta}} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) \lambda \left(\frac{k-1}{2} \right) \left(\frac{k-1}{2} \right)^{-1+\delta} k^{\varepsilon} \right)^{\beta} \sum_{n=\left[\frac{k-1}{2}\right]}^{\infty} n^{-\delta\beta - \varepsilon\beta - 2} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) \lambda \left(\frac{k-1}{2} \right) k^{\varepsilon} \right)^{\beta} \left(\frac{k-1}{2} \right)^{-\beta - \varepsilon\beta - 1} \right)^{\frac{1}{\beta}} \asymp$$

$$\asymp \left(\sum_{k=1}^{\infty} \left(f^* \left(\frac{1}{k} \right) \lambda \left(\frac{k-1}{2} \right) \right)^{\beta} k^{-(\beta+1)} \right)^{\frac{1}{\beta}} \leq$$

$$\leq 2^{1+\frac{1}{\beta}}\Bigg(\sum_{k=1}^{\infty}\left(f^*\left(\frac{1}{k}\right)\frac{\lambda\left(\frac{k-1}{2}\right)\left(\frac{k-1}{2}\right)^{-\delta}}{\left(\frac{k-1}{2}\right)^{-\delta}}\right)^{\beta}\int\limits_{k}^{k+1}\frac{dt}{t^{\beta+1}}\Bigg)^{\frac{1}{\beta}}\leq$$

$$\leq c\Bigg(\sum_{k=1}^{\infty}\int\limits_{k}^{k+1}\left(f^*\left(\frac{1}{t}\right)\lambda(t)t^{-\delta}\right)^{\beta}t^{\delta\beta-\beta-1}dt\Bigg)^{\frac{1}{\beta}}\asymp$$

$$\asymp\Bigg(\int\limits_{1}^{\infty}\left(f^*\left(\frac{1}{t}\right)\frac{\lambda(t)}{t}\right)^{\beta}\frac{dt}{t}\Bigg)^{\frac{1}{\beta}}=c\Bigg(\int\limits_{0}^{1}\left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^{\beta}\frac{dt}{t}\Bigg)^{\frac{1}{\beta}}.$$

We conclude that

$$I_2\leq c\Bigg(\int\limits_{0}^{1}\left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^{\beta}\frac{dt}{t}\Bigg)^{\frac{1}{\beta}}. \tag{3.5}$$

By combining (3.4) and (3.5) we obtain inequality (2.1) and the proof is complete. □

Proof the Proposition 2.1. Let $\lambda(t)$ be from the class B. This means that there exists $\delta > 0$ such that $\lambda(t)t^{-\frac{1}{2}-\delta}$ is an increasing function and $\lambda(t)t^{-1+\delta}$ is a decreasing function. We will use the following well-known lemma of Rudin-Shapiro (see e.g. [2]) for the proof:

Lemma 3.1 There exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$, such that $\varepsilon_n = \pm 1$ for all n and

$$\left|\sum_{n=0}^{N}\varepsilon_n e^{int}\right| < 5\sqrt{N+1}, \tag{4.1}$$

for $t \in [0, 2\pi]$ and $N = 0, 1, \dots$.

Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be the sequence from Lemma 3.1. We will consider $f_k(t)$, $k \in Z$, defined by

$$\sum_{k=1}^{\infty}2^{-\frac{1}{2}k}\frac{1}{k^2}\sum_{n=2^{k-1}}^{2^k-1}\varepsilon_n e^{int} := \sum_{k=1}^{\infty}2^{-\frac{1}{2}k}\frac{1}{k^2}f_k(t). \tag{4.2}$$

According to (4.1) we have that

$$|f_k(t)| \leq \left|\sum_{n=0}^{2^{k-1}-1}\varepsilon_n e^{int}\right| + \left|\sum_{n=0}^{2^k-1}\varepsilon_n e^{int}\right| \leq 10\cdot 2^{\frac{1}{2}k}.$$

By the Weierstrass theorem the series (4.2) converges uniformly on compact intervals and its sum, which we will denote by $f(x)$, is continuous, one periodical and, thus, bounded, i.e. $|f(t)| \leq M$. Hence, obviously, its Fourier coefficients $a_n(f) = \varepsilon_n 2^{-\frac{1}{2}k}\frac{1}{k^2}$, if $2^{k-1} \leq n \leq 2^k$, where $k = 1, 2, \ldots$. We use that $\lambda(t)t^{-1+\delta}$ is a decreasing function and obtain that

$$\left(\int_0^1 \left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^\beta \frac{dt}{t}\right)^{\frac{1}{\beta}} \leq M\left(\int_0^1 \left(\frac{t\lambda\left(\frac{1}{t}\right)\left(\frac{1}{t}\right)^{-1+\delta}}{\left(\frac{1}{t}\right)^{-1+\delta}}\right)^\beta \frac{dt}{t}\right)^{\frac{1}{\beta}} \leq$$

$$\leq M\lambda(1)\left(\int_0^1 t^{(-1+\delta)\beta-1}dt\right) = c < \infty.$$

Let

$$\left(\sum_{n=1}^\infty \left(\overline{|a_n|}\lambda(n)\right)^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} = I.$$

Since

$$\overline{|a_n|} = \frac{1}{n}\sum_{k=1}^n |a_k| \geq \frac{1}{n}\sum_{r=1}^{[\log_2 n]}\sum_{k=2^{r-1}}^{2^r-1}|a_k| \geq \frac{1}{n}\sum_{r=1}^{[\log_2 n]} 2^{-\frac{1}{2}r}\frac{1}{r^2}2^r =$$

$$= \frac{1}{n}\sum_{r=1}^{[\log_2 n]} 2^{\frac{r}{2}}\frac{1}{r^2} \asymp \frac{1}{n}2^{\log_2 \sqrt{n}}\frac{1}{\log_2 n} = \frac{\sqrt{n}}{n\log_2 n},$$

we have that

$$I \geq c\left(\sum_{n=1}^\infty \left(\frac{\lambda(n)}{\sqrt{n}\log_2 n}\right)^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} = c\left(\sum_{n=1}^\infty \left(\frac{\lambda(n)n^{-\frac{1}{2}-\delta}}{n^{-\frac{1}{2}-\delta}\sqrt{n}\log_2 n}\right)^\beta \frac{1}{n}\right)^{\frac{1}{\beta}}.$$

Moreover, in view of the fact that $\{\lambda(n)n^{-\frac{1}{2}-\delta}\}$ is an increasing sequence of n, it yields that

$$c\left(\sum_{n=1}^\infty \frac{1}{n^{1-\delta\beta}(\log_2 n)^\beta}\right)^{\frac{1}{\beta}} = +\infty,$$

and we conclude that also $I = \infty$. The proof is complete. □

Proof the Theorem 2.2. At first we show the regularity of the function $\frac{\left(t\lambda\left(\frac{1}{t}\right)\right)^\beta}{t}$ of generalized space type Λ_β in the form of the following lemma of independent interest:

Lemma 3.1. *Let $\lambda^{-1}(t)$ belong to the class D, then the function $\frac{\left(t\lambda\left(\frac{1}{t}\right)\right)^\beta}{t}$ is a regular function.*

In fact,

$$I=\frac{1}{t}\int\limits_0^t \frac{\left(\tau\lambda\left(\frac{1}{\tau}\right)\right)^\beta}{\tau}d\tau=\frac{1}{t}\int\limits_0^t\left(\lambda\left(\frac{1}{\tau}\right)\right)^\beta\tau^{\beta-1}d\tau,$$

so that, by making a change of variables, we obtain that

$$I=\frac{1}{t}\int\limits_{\frac{1}{t}}^\infty(\lambda(\tau))^\beta\,\tau^{-\beta-1}d\tau=\frac{1}{t}\int\limits_{\frac{1}{t}}^\infty(\lambda(\tau))^\beta\,\tau^{\delta}\tau^{-\delta-\beta-1}d\tau.$$

From this and the fact that $\lambda^{-1}(t)$ belongs in the class D, so that there exists $\delta>0$ that $\lambda^{-1}(t)t^{-\delta}$ is an increasing function and $\lambda^{-1}(t)t^{-1+\delta}$ is a decreasing function, i.e. $\lambda(t)t^{\delta}$ is a decreasing function and $\lambda(t)t^{1-\delta}$ is an increasing function. Then we have the following estimate

$$I\leq\frac{1}{t}\lambda\left(\frac{1}{t}\right)^\beta\left(\frac{1}{t}\right)^\delta\int\limits_{\frac{1}{t}}^\infty\tau^{-\delta-\beta-1}=C(\beta,\delta)\frac{\left(t\lambda\left(\frac{1}{t}\right)\right)^\beta}{t}$$

The Lemma is proved and we return to the proof of the Theorem. According to Lemma 3.2 and Theorem 2.4.12 (ii) in the book [14] the following equality holds:

$$\Lambda_\beta\left(\lambda\right)=\left(\Lambda_{\beta'}\left((t\lambda)^{-1}\right)\right)',\ \text{ if } 1<\beta<\infty.$$

Hence, from the duality representation of the norm of a function f in the space Λ_β (see [14]) we obtain that

$$\|f\|_{\Lambda_\beta}=\sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1}\int\limits_0^1 f(x)\cdot g(x)dx$$

Now we use the Parseval's equality and find that

$$\|f\|_{\Lambda_\beta}=\sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1}\sum_{n=1}^\infty a_nb_n,$$

and then, by using Abel's transformation, we have that

$$\|f\|_{\Lambda_\beta}=\sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1}\sum_{n=1}^{N-1}(a_n-a_{n+1})\sum_{m=1}^n b_m+a_N\sum_{m=1}^N b_m.$$

Next we note that

$$a_N\sum_{m=1}^N b_m\leq\lambda(N)a_nN\lambda^{-1}(N)\frac{1}{N}\left|\sum_{m=1}^N b_m\right|\leq$$

$$\leq\lambda(N)a_n\|g\|_{\Lambda_\infty((t\lambda)^{-1})}\leq c_1\lambda(N)a_n\|g\|_{\Lambda_{\beta\left((t\lambda)^{-1}\right)}}.$$

Hence, the sequence $\left\{a_N \sum_{m=1}^N b_m\right\}_{N=1}^\infty$ is bounded so, in particular, we can pass to the limit $N \to \infty$. Therefore, by using our assumption $\lim_{N\to\infty} \lambda(N)a_N = 0$, we obtain that

$$\|f\|_{\Lambda_\beta} = \sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1} \sum_{n=1}^\infty (a_n - a_{n+1}) \sum_{m=1}^n b_m.$$

Since $\overline{b}_n = \sup_{r\geq n} \frac{1}{r}\left|\sum_{m=1}^r b_m(f)\right|$, we thus obtain that

$$I \leq \sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1} \sum_{n=1}^\infty |n\Delta a_n|\overline{b_n}.$$

Therefore, by using Hölder's inequality, we get that

$$\|f\|_{\Lambda_\beta} \leq \sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1} \left(\sum_{n=1}^\infty \left(|n\Delta a_n|\lambda(n)n^{-1+\frac{1}{\beta'}}\right)^\beta\right)^{\frac{1}{\beta}} \times$$
$$\times\left(\sum_{n=1}^\infty \left(\overline{b_n}\lambda^{-1}(n)n^{1-\frac{1}{\beta'}}\right)^{\beta'}\right)^{\frac{1}{\beta'}} =$$
$$= \sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1} \left(\sum_{n=1}^\infty (|n\Delta a_n|\lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} \left(\sum_{n=1}^\infty \left(\overline{b_n}\lambda^{-1}(n)n\right)^{\beta'} \frac{1}{n}\right)^{\frac{1}{\beta'}}.$$

Further, by applying the inequality (2.1) from Theorem 2.1, we obtain the ciaimed estimate:

$$\|f\|_{\Lambda_\beta} \leq c \sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1} \left(\sum_{n=1}^\infty (|n\Delta a_n|\lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} \times$$
$$\times\left(\int_0^1 \left(g^*(t)t\frac{\lambda^{-1}\left(\frac{1}{t}\right)}{t}\right)^{\beta'} \frac{dt}{t}\right)^{\frac{1}{\beta'}} =$$
$$= c \sup_{\|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})}=1} \left(\sum_{n=1}^\infty (|n\Delta a_n|\lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} \|g\|_{\Lambda_{\beta'}((t\lambda)^{-1})} =$$
$$= c\left(\sum_{n=1}^\infty (|n\Delta a_n|\lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}}.$$

The proof is complete. □

Proof the Theorem 2.3. Since B is a subclass of D the proof in one direction follows from our Theorem 2.1. Since a regular system is bounded we can use Theorem 1 in [4] (see also [5]) to obtain the estimate in the other direction. The proof is complete. □

4. Concluding Remarks and Examples

Remark 4.1. The classes B_δ and D_δ are just special cases of the more general classes Q_α^β studied in [10] in connection to interpolation theory (We say that $\lambda \in Q_\alpha^\beta$ if, for some $\delta > 0$, $\lambda(t)t^{-\alpha-\delta}$ is an increasing function and $\lambda(t)t^{-\beta+\delta}$ is a decreasing function). In particular, it was proved there that λ in the class Q_α^β in fact is equivalent with a function λ_0 with upper and lower indices α and β, respectively, and also equivalent to some other classes of index type used in interpolation theory (e.g. the Peetre-Gustavsson $\pm$ class.)

Remark 4.2. The assumptions in our theorems can obviously be weakened on some points. For example in Theorem 2.1 we only need to assume that $\lambda(t)$ is equivalent to a function from the class D. For example in the definition of the class D_δ we only need to assume that $\lambda(t)t^{-\delta} \leq c_0\lambda(s)s^{-\delta}$ and $\lambda(s)s^{-1+\delta} \leq c_1\lambda(t)t^{-1+\delta}$ for $t \leq s$ and some positive constants c_0 and c_1. Hence, according to our Remark 4.1 this gives us the possibility to formulate our result in terms of indices.

We also present some more examples to illustrate the importance of the concept of generalized monotone sequences.

Example 4.1. Let $k \in N$ and consider

$$a_k = \begin{cases} \frac{1}{k}, & \text{if} \quad 2^{n-1} \leq k < 2^n, n \text{ is even}, \\ 0, & \text{if} \quad 2^{n-1} \leq k < 2^n, n \text{ is odd}, \end{cases}$$

$$b_k = \begin{cases} \frac{1}{k}, & \text{if} \quad 2^{n-1} \leq k < 2^n, n \text{ is odd}, \\ 0, & \text{if} \quad 2^{n-1} \leq k < 2^n, n \text{ is even}, n \in N. \end{cases}$$

Then $c = \{a_k + ib_k\}_{k=1}^\infty$ is generalized monotone but each of the sequences $\{a_k\}$ and $\{b_k\}$ is not a generalized monotone sequence.

Example 4.2. Let

$$a_k = \frac{(-1)^{k+1}}{k^\alpha}, \quad k \in N.$$

If $\alpha \geq 1$, then the sequence $a = \{a_k\}$ is generalized monotone but if $\alpha < 1$, then it is not generalized monotone.

In fact

$$\frac{|a_n|}{\frac{1}{n}\left|\sum_{k=1}^n a_k\right|} = \frac{\frac{1}{n^\alpha}}{\frac{1}{n}\left|\sum_{k=1}^n \frac{(-1)^{k+1}}{k^\alpha}\right|} =$$

$$= \frac{n^{1-\alpha}}{\left|\sum_{k=1}^n \frac{(-1)^{k+1}}{k^\alpha}\right|} = B_n.$$

If $\alpha < 1$, then $B_n \to \infty$ when $n \to \infty$ and the condition (2.2) is not fulfilled. If $\alpha \geq 1$ then $\lim_{n\to\infty} B_n$, exists and the sequence $\{B_n\}$ is limited, i.e. $0 < B_n \leq M$. Hence, the sequence $\{a_k\}$ is generalized monotone.

Remark 4.3. It seems to be possible to study some corresponding questions for Fourier transforms and other related integral transforms. The present authors aim to develop this idea in a forthcoming paper.

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