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# ON SUMMABILITY OF THE FOURIER COEFFICIENTS IN BOUNDED ORTHONORMAL SYSTEMS FOR FUNCTIONS FROM SOME LORENTZ TYPE SPACES

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**Abstract.** We denote by $\Lambda_\beta(\lambda)$, $\beta > 0$, the Lorentz space equipped with the (quasi) norm

$$\|f\|_{\Lambda_\beta(\lambda)} := \left( \int_0^1 \left( f^*(t) t \lambda \left( \frac{1}{t} \right) \right)^\beta \frac{dt}{t} \right)^{\frac{1}{\beta}}$$

for a function $f$ on [0,1] and with $\lambda$ positive and equipped with some additional growth properties. Some estimates of this quantity and some corresponding sums of Fourier coefficients are proved for the case with a general orthonormal bounded system.

## 1 Introduction

Let $f$ be a measurable function on a measure space $(\Omega, \mu)$, where $\mu$ is an additive positive measure.

The distribution function $m(\sigma, f)$ and the nonincreasing rearrangement $f^*$ of a function $f$ are defined as follows:

$$m(\sigma, f) := \mu \{x \in \Omega : |f(x)| > \sigma\},$$

$$f^*(t) := \inf \{\sigma : m(\sigma, f) \leq t\}.$$

Let $1 \leq p \leq \infty$ and $0 < q \leq \infty$. The Lorentz space $L_{pq}$ consists of all functions $f$ satisfying

$$\|f\|_{L_{pq}} := \left( \int_0^\infty t^{\frac{q}{p}-1} (f^*(t))^q dt \right)^{\frac{1}{q}} < \infty. \quad (1)$$

Note that for the case $p = q$ the $L_{pq}$ spaces coincide with the usual $L_p$ spaces equipped with the norms $\|f\|_{L_p}$(quasinorms for $0 < p < 1$, see [7] and [6]).

Let $0 < \beta \leq \infty$ and let $\lambda$ be a nonnegative function on $[0, \infty]$. Some generalized Lorentz spaces $\Lambda_q(\varphi)$, which can be obtained by replacing the function $t^{\frac{1}{p}-\frac{1}{q}}$ in (1) by an some positive weight function $\varphi(t)$, are frequently studied in the literature. In this paper we shall use a special case introduced by L.-E. Persson in the Ph.D thesis from 1974 (see [10] and also [3]) in connection to Fourier series.

The generalized Lorentz space $\Lambda_\beta(\lambda) \equiv \Lambda_\beta(\lambda)[0,1]$ consists of the functions $f$ on $[0,1]$ such that: $\|f\|_{\Lambda_\beta(\lambda)} < \infty$, where

$$\|f\|_{\Lambda_\beta(\lambda)} := \begin{cases} \left(\int_0^1 \left(f^*(t)t\lambda(\frac{1}{t})\right)^\beta \frac{dt}{t}\right)^{\frac{1}{\beta}} & \text{for } \ 0<\beta<\infty, \\ \sup\limits_{0\le t\le 1} f^*(t)t\lambda(\frac{1}{t}) & \text{for } \ \beta=\infty. \end{cases}$$

Let the function $f$ be periodic with period 1 and integrable on $[0,1]$ and let $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system on $[0,1]$. The numbers

$$a_n = a_n(f) = \int_0^1 f(x)\overline{\varphi_n(x)}dx, \qquad n \in \mathbb{N},$$

are called the Fourier coefficients of the function $f$ with respect to the system $\Phi = \{\varphi_n\}_{n=1}^{\infty}$.

We also remark that there are many relations between summability of Fourier coefficients and integrability of the corresponding functions e.g. the following two-sided ones for the trigonometrical system:

$$\|f\|^p_{L_p[0,1]} \le c_1 \sum_{k=1}^{\infty} k^{p-2}|a_k|^p, \qquad \text{if} \quad 2\le p<\infty, \tag{2}$$

$$\|f\|^p_{L_p[0,1]} \ge c_2 \sum_{k=1}^{\infty} k^{p-2}|a_k|^p, \qquad \text{if} \quad 1<p\le 2. \tag{3}$$

The inequalities (2) and (3) are the classical ones, which can be found already in the Hardy-Littlewood-Pólya book [1]. These inequalities were early generalized to hold also for Lorentz spaces by Stein [15] and for the more general Lorentz spaces $\Lambda_\beta(\lambda)$ by L.-E. Persson(see [10] and [3]):

**Theorem 1.** *Let* $0<\beta<\infty$ *and* $\Phi = \left\{e^{2\pi ikt}\right\}_{k=-\infty}^{+\infty}$ *be a trigonometrical system.*
*a) If there exists a positive number* $\delta > 0$ *satisfying such conditions like:* $\lambda(t)t^{-\delta}$ *is an increasing function of* $t$ *and* $\lambda(t)t^{-\left(\frac{1}{2}-\delta\right)}$ *is a decreasing function of* $t$*, then*

$$\left(\sum_{n=1}^{\infty}(a_n^*\lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} \le c_1\|f\|_{\Lambda_\beta(\lambda)}.$$

*b) If there exists a positive number* $\delta>0$ *satisfying such conditions like:* $\lambda(t)t^{-\frac{1}{2}-\delta}$ *is an increasing function of* $t$ *and* $\lambda(t)t^{-1+\delta}$ *is a decreasing function of* $t$*, then*

$$\|f\|_{\Lambda_\beta(\lambda)} \le c_2 \left(\sum_{n=1}^{\infty}(a_n^*\lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}},$$

*where* $\{a_n^*\}_{n=1}^{\infty}$ *is the nonincreasing rearrangement of the sequence* $\{|a_k|\}_{k=1}^{\infty}$ *of Fourier coefficients of* $f$ *with respect to the system* $\Phi$.

The aim of this paper is to derive some analogues of the inequalities (2) and (3) both in the case of bounded orthonormal systems $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ and for generalized Lorentz spaces of type $\Lambda_\beta(\lambda)$.

**Conventions.** The letter $c$ $(c_1, c_2, etc.)$ means a constant which does not dependent on the involved functions and it can be different in different occurences. Moreover, for $C, D > 0$ the notation $C \sim D$ means that there exist positive constants $a_1$ and $a_2$ such that $a_1 D \leq C \leq a_2 D$.

## 2 The main result

Let $\delta > 0$ and $\lambda(t)$ be a nonnegative function on $[1, \infty)$. We define the following classes (see also [12]):

$$A_\delta = \{\lambda(t) : \lambda(t)t^{-\delta} \text{ is an increasing function and } \lambda(t)t^{-\left(\frac{1}{2}-\delta\right)} \text{is a decreasing function}\Big\},$$

$$B_\delta = \{\lambda(t) : \lambda(t)t^{-\frac{1}{2}-\delta} \text{ is an increasing function and } \lambda(t)t^{-1+\delta} \text{ is a decreasing function}\big\}.$$

Then the classes $A$ and $B$ are defined as follows:

$$A = \cup_{\delta>0} A_\delta, \quad B = \cup_{\delta>0} B_\delta.$$

*In the sequel we denote by $\Phi = \{\varphi_n\}_1^{\infty}$ a bounded orthonormal system, i.e., $|\varphi_n(t)| \leq$ $M$,*
$t \in [0,1]$, $n \in \mathbb{N}$. Our result reads:

**Theorem 2.** *Let $0 < \beta \leq \infty$, and assume that the orthonormal system $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is bounded.*
*(a) If $\lambda(t)$ belongs to the class $A$, then*

$$\left(\sum_{n=1}^{\infty} (a_n^* \lambda(n))^\beta \frac{1}{n}\right)^{\frac{1}{\beta}} \leq c_1 \|f\|_{\Lambda_\beta(\lambda)}, \tag{4}$$

*where $\{a_n^*\}_{n=1}^{\infty}$ is the nonincreasing rearrangement of the sequence $\{a_k\}_{k=1}^{\infty}$ of Fourier coefficients of $f$ with respect to the system $\Phi$.*
*(b) If $\lambda(t)$ belongs to the class $B$ and $f \overset{a.e.}{=} \sum_{n=1}^{\infty} a_n \varphi_n$, then*

$$\|f\|_{\Lambda_\beta(\lambda)} \leq c_2 \left(\sum_{n=1}^{\infty} (a_n^* \lambda(n)^\beta \frac{1}{n}\right)^{\frac{1}{\beta}}. \tag{5}$$

*Here the constants $c_1$ and $c_2$ don't depend on $f$.*

**Proof.** (a) Let $\lambda(t)$ belongs to the class $A$. This means that there exists $\delta > 0$ such that: $\lambda(t)t^{-\delta}$ is an increasing function and $\lambda(t)t^{-\left(\frac{1}{2}-\delta\right)}$ is a decreasing function. Suppose that the function $f$ satisfies the condition:

$$\left(\int_0^1 \left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^\beta \frac{dt}{t}\right)^{\frac{1}{\beta}} < \infty.$$

Let $f = f_0 + f_1$, where $f_0$ and $f_1$ are defined later on. By using the inequalities

$$\|a\|_{l_{2\infty}} \leq c_1 \|f\|_{L_{21}},$$

$$\|a\|_{l_\infty} \leq c_2 \|f\|_{L_1} \qquad \text{and}$$

$$a_n^*(f) \leq a^*_{\left[\frac{n}{2}\right]}(f_0) + a^*_{\left[\frac{n}{2}\right]}(f_1), \qquad n = 1, 2, ...,$$

we estimate $a_n^*(f)$ from above as follows:

$$a_n^*(f) \leq a^*_{\left[\frac{n}{2}\right]}(f_0) + \left(\frac{2}{n}\right)^{\frac{1}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}} a^*_{\left[\frac{n}{2}\right]}(f_1) \leq$$

$$\leq c_3 \left(\int_0^1 f_0^*(t)dt + \frac{1}{n^{\frac{1}{2}}} \int_0^1 t^{-\frac{1}{2}} f_1^*(t)dt\right).$$

Define the functions $f_0$ and $f_1$ in the following way:

$$f_0(t) = \begin{cases} f(t) - f^*(\frac{1}{n}), & \text{as} \ \ |f(t)| \geq f^*(\frac{1}{n}) \\ 0, & \text{as} \ \ |f(t)| < f^*(\frac{1}{n}), \end{cases} \tag{6}$$

$$f_1(t) = \begin{cases} f^*(\frac{1}{n}), & \text{as} \ \ |f(t)| > f^*(\frac{1}{n}) \\ f(t), & \text{as} \ \ |f(t)| \leq f^*(\frac{1}{n}). \end{cases} \tag{7}$$

Now, by using (6) and (7) we obtain that

$$\int_0^1 f_0^*(t)dt = \int_0^{\frac{1}{n}} \left(f^*(t) - f^*\left(\frac{1}{n}\right)\right) dt = \int_0^{\frac{1}{n}} f^*(t)dt - \frac{f^*(\frac{1}{n})}{n}, \tag{8}$$

$$\frac{1}{n^{\frac{1}{2}}} \int_0^1 t^{-\frac{1}{2}} f_1^*(t)dt = \frac{1}{n^{\frac{1}{2}}} \left(\int_0^{\frac{1}{n}} t^{-\frac{1}{2}} f^*\left(\frac{1}{n}\right) dt + \int_{\frac{1}{n}}^1 t^{-\frac{1}{2}} f^*(t)dt\right) = \tag{9}$$

$$= \frac{2f^*(\frac{1}{n})}{n} + \frac{1}{n^{\frac{1}{2}}} \int_{\frac{1}{n}}^1 t^{-\frac{1}{2}} f^*(t)dt.$$

According to (8) and (9) we find that

$$\int_0^{\frac{1}{n}} f^*(t)dt - \frac{f^*(\frac{1}{n})}{n} + \frac{2f^*(\frac{1}{n})}{n} + \frac{1}{n^{\frac{1}{2}}} \int_{\frac{1}{n}}^1 t^{-\frac{1}{2}} f^*(t)dt =$$

$$= \int_0^{\frac{1}{n}} f^*(t)dt + \frac{f^*(\frac{1}{n})}{n} + \frac{1}{n^{\frac{1}{2}}} \int_{\frac{1}{n}}^1 t^{-\frac{1}{2}} f^*(t)dt \leq$$

$$\leq 2\int_0^{\frac{1}{n}} f^*(t)dt + \frac{1}{n^{\frac{1}{2}}}\int_{\frac{1}{n}}^1 t^{-\frac{1}{2}} f^*(t)dt,$$

hence, by making a change of variables, we get that

$$2\int_n^{\infty} f^*\left(\frac{1}{n}\right)\frac{dt}{t^2} + \frac{1}{n^{\frac{1}{2}}}\int_1^n f^*\left(\frac{1}{t}\right)\frac{dt}{t^{-\frac{1}{2}+2}} \sim$$

$$\sim \left(\sum_{k=n}^{\infty} f^*\left(\frac{1}{k}\right)k^{-2} + \frac{1}{n^{\frac{1}{2}}}\sum_{k=1}^{n} f^*\left(\frac{1}{k}\right)\frac{1}{k^{-\frac{1}{2}+2}}\right). \tag{10}$$

In view of (10) and Minkowski's inequality we have that

$$I := \left(\sum_{n=1}^{\infty}\left(a_n^*\lambda(n)\right)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} \leq$$

$$\leq c_4\left(\sum_{n=1}^{\infty}\left(\lambda(n)\sum_{k=n}^{\infty} f^*\left(\frac{1}{k}\right)k^{-2} + \frac{\lambda(n)}{n^{\frac{1}{2}}}\sum_{k=1}^{n} f^*\left(\frac{1}{k}\right)\frac{1}{k^{-\frac{1}{2}+2}}\right)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} \leq$$

$$\leq c_5\left(\sum_{n=1}^{\infty}\left(\lambda(n)\sum_{k=n}^{\infty} f^*\left(\frac{1}{k}\right)k^{-2}\right)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} +$$

$$+\left(\sum_{n=1}^{\infty}\left(\frac{\lambda(n)}{n^{\frac{1}{2}}}\sum_{k=1}^{n} f^*\left(\frac{1}{k}\right)\frac{1}{k^{-\frac{1}{2}+2}}\right)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} :=$$

$$:= c_5\left(I_1 + I_2\right). \tag{11}$$

Firstly, we consider $I_1$. Let $\varepsilon$ be such that $-\frac{1}{\beta} - 1 < \varepsilon < \delta - \frac{1}{\beta} - 1$. We use Hölder's inequality and since $-\frac{1}{\beta} - 1 < \varepsilon$, we have that

$$I_1 \leq c_6\left(\sum_{n=1}^{\infty}\left(\lambda(n)\left(\sum_{k=n}^{\infty}\left(f^*\left(\frac{1}{k}\right)k^{\varepsilon}\right)^{\beta}\right)^{\frac{1}{\beta}}\cdot\left(\sum_{k=n}^{\infty}\left(\frac{1}{k^{\varepsilon+2}}\right)^{\beta'}\right)^{\frac{1}{\beta'}}\right)^{\beta}\frac{1}{n}\right)^{\frac{1}{\beta}} \sim$$

$$\sim \left(\sum_{k=1}^{\infty}\left(f^*\left(\frac{1}{k}\right)k^{\varepsilon}\right)^{\beta}\sum_{n=1}^{k}\lambda^{\beta}(n)n^{\frac{\beta}{\beta'}-\beta(\varepsilon+2)}\frac{1}{n}\right)^{\frac{1}{\beta}}.$$

Hence, by using the fact that $\lambda(t)t^{-\delta}$ is an increasing function, we obtain that

$$I_1 \leq c_7\left(\sum_{k=1}^{\infty}\left(f^*\left(\frac{1}{k}\right)\frac{\lambda(k)}{k^{\delta}}k^{\varepsilon}\right)^{\beta}\sum_{n=1}^{k} n^{(\delta-\varepsilon-1)\beta-2}\right)^{\frac{1}{\beta}}.$$

Thus, taking into account that $\varepsilon < \delta - \frac{1}{\beta} - 1$, and by using some obvious estmates we find that

$$I_1 \le c_8 \left( \sum_{k=1}^{\infty} \left( f^*\left(\frac{1}{k}\right) \frac{\lambda(k)}{k} \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}} \le$$

$$\le c_9 \left( \sum_{k=1}^{\infty} \left( f^*\left(\frac{1}{k}\right) \lambda(k) \right)^{\beta} \int_k^{k+1} \frac{dt}{t^{1+\beta}} \right)^{\frac{1}{\beta}} \le$$

$$\le c_{10} \left( \sum_{k=1}^{\infty} \int_k^{k+1} \left( \frac{f^*\left(\frac{1}{t}\right) \lambda(t) t^{-\delta}}{t^{-\delta+1}} \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} = c_{10} \left( \int_1^{\infty} \left( f^*\left(\frac{1}{t}\right) \frac{\lambda(t)}{t} \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}}.$$

Consequently, we get that

$$I_1 \le c_{10} \left( \int_0^1 \left( f^*(t) t \lambda\left(\frac{1}{t}\right) \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}}. \tag{12}$$

Let us now estimate $I_2$. Choose $\varepsilon$ such that $\frac{1}{2} + \frac{1}{\beta} < \varepsilon < \frac{1}{2} + \frac{1}{\beta} + \delta$. By using Hölder's inequality and that $\frac{1}{2} + \frac{1}{\beta} < \varepsilon$, we find that

$$I_2 = \left( \sum_{n=1}^{\infty} \left( \frac{\lambda(n)}{n^{\frac{1}{2}}} \sum_{k=1}^{n} f^*\left(\frac{1}{k}\right) \frac{1}{k^{-\frac{1}{2}+2}} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \le$$

$$\le c_{11} \left( \sum_{n=1}^{\infty} \left( \frac{\lambda(n)}{n^{\frac{1}{2}}} \left( \sum_{k=1}^{n} \left( f^*\left(\frac{1}{k}\right) k^{-\varepsilon} \right)^{\beta} \right)^{\frac{1}{\beta}} \left( \sum_{k=1}^{n} \frac{1}{k^{(\frac{3}{2}-\varepsilon)\beta'}} \right)^{\frac{1}{\beta'}} \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}} \sim$$

$$\sim \left( \sum_{n=1}^{\infty} \lambda^{\beta}(n) \cdot n^{(\varepsilon-1)\beta-2} \sum_{k=1}^{n} \left( f^*\left(\frac{1}{k}\right) k^{-\varepsilon} \right)^{\beta} \right)^{\frac{1}{\beta}} =$$

$$= c_{12} \left( \sum_{k=1}^{\infty} \left( f^*\left(\frac{1}{k}\right) k^{-\varepsilon} \right)^{\beta} \sum_{n=k}^{\infty} \frac{\lambda^{\beta}(n)}{n^{\left(\frac{1}{2}-\delta\right)\beta}} n^{\left(\frac{1}{2}-\delta\right)\beta} n^{(\varepsilon-1)\beta-2} \right)^{\frac{1}{\beta}}.$$

Hence, by using the fact that $\lambda(t) t^{-\left(\frac{1}{2}-\delta\right)}$ is a decreasing function and taking into account that $\varepsilon < \frac{1}{2} + \frac{1}{\beta} + \delta$ we obtain that

$$I_2 \le c_{12} \left( \sum_{k=1}^{\infty} \left( f^*\left(\frac{1}{k}\right) k^{-\varepsilon} \frac{\lambda(k)}{k^{\frac{1}{2}-\delta}} \right)^{\beta} \sum_{n=k}^{\infty} n^{(\varepsilon-\delta-\frac{1}{2})\beta-2} \right)^{\frac{1}{\beta}} \le$$

$$\le c_{13} \left( \sum_{k=1}^{\infty} \left( f^*\left(\frac{1}{k}\right) \frac{\lambda(k)}{k} \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}} \le$$

$$\leq c_{13}\left(\int_0^1\left(f^*(t)\lambda\left(\frac{1}{t}\right)t\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}. \tag{13}$$

Thus, by combinig (11), (12) and (13) we prove (4).

(b) Let $\lambda(t)$ belongs to the class $B$. This means that there exists $\delta>0$ such that: $\lambda(t)t^{-\frac{1}{2}-\delta}$ is an increasing function and $\lambda(t)t^{-1+\delta}$ is a decreasing function. Let $a=\{a_n\}_{n=1}^{\infty}$, $a=a_0+a_1$, $a_0=\{a_n^0\}_{n=1}^{\infty}$, $a_1=\{a_n^1\}_{n=1}^{\infty}$, where

$$a_n^0=\begin{cases} a_n, & \text{when } |a_n|\geq a^*_{\left[\frac{1}{t}\right]} \\ 0, & \text{when } |a_n|< a^*_{\left[\frac{1}{t}\right]}, \end{cases} \tag{14}$$

and

$$a_n^1=a_n-a_n^0. \tag{15}$$

Then

$$f^*(t)=\left(\sum_{n=1}^{\infty}a_n^0\varphi_n\right)^*+\left(\sum_{n=1}^{\infty}a_n^1\varphi_n\right)^*\leq$$

$$\leq\left(M\sum_{n=1}^{\infty}\left|a_n^0\right|\right)^*+\left(M\sum_{n=1}^{\infty}\left|a_n^1\right|\right)^*.$$

Hence, according to (14) and (15), it yields that

$$f^*(t)\leq M\sum_{n=1}^{\left[\frac{1}{t}\right]}a_n^*+M\sum_{n=\left[\frac{1}{t}\right]+1}^{\infty}a_n^*.$$

By using this information and Minkowski's inequality we find that

$$I_0:=\left(\int_0^1\left(f^*(t)t\lambda\left(\frac{1}{t}\right)\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}\leq$$

$$\leq M\left(\int_0^1\left(\left(\sum_{n=1}^{\left[\frac{1}{t}\right]}a_n^*+\sum_{n=\left[\frac{1}{t}\right]}^{\infty}a_n^*\right)t\lambda\left(\frac{1}{t}\right)\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}=$$

$$\leq M\left(\int_0^1\left(t\lambda\left(\frac{1}{t}\right)\sum_{n=1}^{\left[\frac{1}{t}\right]}a_n^*\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}+M\left(\int_0^1\left(t\lambda\left(\frac{1}{t}\right)\sum_{n=\left[\frac{1}{t}\right]}^{\infty}a_n^*\right)^{\beta}\frac{dt}{t}\right)^{\frac{1}{\beta}}:=$$

$$:=M\left(I_1+I_2\right).$$

We consider first $I_1$. Choose $\varepsilon$ such that $-\frac{1}{\beta}<\varepsilon<-\frac{1}{\beta}+\delta$. Since $\lambda\left(t\right)t^{-1+\delta}$ is a decreasing function it yields that

$$I_1 = \left( \int_0^1 \left( \lambda(\frac{1}{t}) t \sum_{n=1}^{[\frac{1}{t}]} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} =$$

$$= \left( \int_0^1 \left( t \frac{\lambda\left(\frac{1}{t}\right)\left(\frac{1}{t}\right)^{-1+\delta}}{\left(\frac{1}{t}\right)^{-1+\delta}} \sum_{n=1}^{[\frac{1}{t}]} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c_1 \left( \int_0^1 \left( t^{\delta} \sum_{n=1}^{[\frac{1}{t}]} \lambda\left(n\right) n^{-1+\delta} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} =$$

$$= c_2 \left( \int_1^{\infty} \left( t^{-\delta} \sum_{n=1}^{[t]} \lambda\left(n\right) n^{-1+\delta} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} \sim$$

$$\sim \left( \sum_{k=1}^{\infty} \left( k^{-\delta} \sum_{n=1}^{[k]} \lambda\left(n\right) n^{-1+\delta} a_n^* \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}}.$$

By now using Hölder's inequality and the fact that $-\frac{1}{\beta} < \varepsilon < -\frac{1}{\beta} + \delta$, we derive that

$$I_1 \leq c_2 \left( \sum_{k=1}^{\infty} \left( k^{-\delta} \left( \sum_{n=1}^{k} \left(\lambda\left(n\right) n^{\varepsilon} a_n^*\right)^{\beta} \right)^{\frac{1}{\beta}} \left( \sum_{n=1}^{k} n^{(-1+\delta-\varepsilon)\beta'} \right)^{\frac{1}{\beta'}} \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}} \sim$$

$$\sim \left( \sum_{k=1}^{\infty} k^{-\delta\beta} k^{(-1+\delta-\varepsilon)\beta + \frac{\beta}{\beta'}} \frac{1}{k} \sum_{n=1}^{k} \left(\lambda\left(n\right) n^{\varepsilon} a_n^*\right)^{\beta} \right)^{\frac{1}{\beta}} =$$

$$= c_3 \left( \sum_{n=1}^{\infty} \left(\lambda\left(n\right) n^{\varepsilon} a_n^*\right)^{\beta} \sum_{k=n}^{\infty} k^{-\varepsilon\beta-2} \right)^{\frac{1}{\beta}} \sim \left( \sum_{n=1}^{\infty} \left(\lambda\left(n\right) a_n^*\right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}}.$$

Hence,

$$I_1 \leq c_4 \left( \sum_{n=1}^{\infty} \left(\lambda\left(n\right) a_n^*\right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}}. \tag{16}$$

Our next aim is to derive a similar estimate for $I_2$. Choose $\varepsilon > 0$ such that $-\delta - \frac{1}{\beta} + \frac{1}{2} < \varepsilon < -\frac{1}{\beta}$. Since $\lambda(t) t^{-\frac{1}{2}-\delta}$ is an increasing function, we obtain the following estimates:

$$I_2 = \left( \int_0^1 \left( t \lambda\left(\frac{1}{t}\right) \sum_{n=[\frac{1}{t}]}^{\infty} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} =$$

$$= \left( \int_0^1 \left( \left( \sum_{n=[\frac{1}{t}]}^{\infty} a_n^* \right) \left( \frac{1}{t} \right)^{-\frac{1}{2}-\delta} \left( \frac{1}{t} \right)^{\frac{1}{2}+\delta} \lambda \left( \frac{1}{t} \right) t \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} \leq$$

$$\leq c_5 \left( \int_0^1 \left( t^{\frac{1}{2}-\delta} \sum_{n=[\frac{1}{t}]}^{\infty} \lambda(n) \, n^{-\frac{1}{2}-\delta} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} =$$

$$= c_5 \left( \int_1^{\infty} \left( t^{-\frac{1}{2}+\delta} \sum_{n=[t]}^{\infty} \lambda(n) \, n^{-\frac{1}{2}-\delta} a_n^* \right)^{\beta} \frac{dt}{t} \right)^{\frac{1}{\beta}} \sim$$

$$\sim \left( \sum_{k=1}^{\infty} \left( k^{-\frac{1}{2}+\delta} \sum_{n=k}^{\infty} \lambda(n) \, n^{-\frac{1}{2}-\delta} a_n^* \right)^{\beta} \frac{1}{k} \right)^{\frac{1}{\beta}}.$$

Hence, we have that

$$I_2 \leq c_6 \left( \sum_{n=1}^{\infty} \left( a_n^* \lambda(n) \right)^{\beta} \frac{1}{n} \right)^{\frac{1}{\beta}}. \tag{17}$$

By combining (16) with (17), we obtain (5) and the proof is complete. □

**Remark.** *An analogous theorem was proved in 1974 by L.-E. Persson, under the assumption that* $\Phi = \left\{ e^{2\pi ikx} \right\}_{k=-\infty}^{+\infty}$ *is a trigonometrical system and* $\beta < \infty$ *(see [10] and also [3]).*

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