THE METHOD OF NET SPACES

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Dedicated to Alois Kufner on the occasion of his 70th birthday

Abstract. In this paper applications of net spaces in different problems are demonstrated. Upper and lower estimates for any function from the L_p space is established in terms of the Fourier coefficients of trigonometric series. The upper and lower estimates of the norm of a convolution and of Fourier multipliers in the L_p spaces are proved. Properties of the norm of Fourier series in Lorentz spaces are studied.

Properties of Fourier coefficients of functions, Fourier transform of functions, norm and spectrum of integral operators depend not only on properties of smoothness and metrics but also on the "geometrical" properties of corresponding functions and kernels. Here we mean such properties as positional relationship of singularities, their sign orientation. Most of known spaces are not "sensitive" to the pointed out property.

In this paper the net space $N_{pq}(M)$ is considered. This space (unlike the Lebesgue and Lorentz spaces) is "sensitive" to distribution of singularities of functions and determines a certain method. The purpose of this paper is to show how methods of net spaces can be used in the different problems. More detailed description of the method you can find in the papers [6]–[9].

1. Net spaces

Let μ be n-dimensional Lebesgue measure in $\mathbb{R}^n$ and let M be a fixed family of finite-measure subsets of $\mathbb{R}^n$. We call M a net in what follows. For a function f defined and integrable on each e in M let

$$\overline{f}(t, M) = \sup_{\substack{e \in M \\ |e| > t}} \frac{1}{|e|} \left| \int_e f(x) \, \mathrm{d}\mu \right|,$$

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where the least upper bound is taken over all $e \in M$, of measure $|e| \stackrel{\text{def}}{=} \mu e > t$, $t \in (0, \infty)$. If $\sup\{|e| : e \in M\} = \alpha < \infty$ and $t > \alpha$, then we set $\overline{f}(t, M) = 0$. We call $\overline{f}(t, M)$ the mean function of f with respect to the net M.

Let $N_{p,q}(M)$, $0 < p, q \le \infty$, be the set of functions f such that

$$\|f\|_{N_{pq}(M)} = \left(\int_0^\infty \left(t^{1/p} \overline{f}(t, M)\right)^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty$$

if $q < \infty$, or

$$\|f\|_{N_{p\infty}(M)} = \sup_{e \in M} \frac{1}{|e|^{1-1/p}} \left| \int_e f(x)\,\mathrm{d}\mu \right| < \infty$$

if $q = \infty$.

R e m a r k 1. a) The net space $N_{pq}(M)$ is quasinormed (and even normed for $q \ge 1$) because it is the quotient space by the kernel $\{f : \int_e f(x)\,\mathrm{d}x = 0,$ $e \in M\}$.

b) The spaces $N_{pq}(M)$ are not symmetric in general.

E x a m p l e 1. Assume that $1 < p, q < \infty$ and let M be the set of closed intervals in $\mathbb{R}$. Then the function $f(x) = (-1)^{[x]}$ (here $[x]$ is the integer part of x) belongs to $N_{p,q}(M)$, but $|f| \notin N_{p,q}(M)$.

R e m a r k 2. The spaces $N_{pq}(M)$ are to a certain extent sensitive to the distribution of oscillations and singularities of functions.

E x a m p l e 2. Let M be the set of closed intervals in $\mathbb{R}$ and let $f_\Omega(x)$ be the characteristic function of the set $\Omega = \bigcup_{k=1}^\infty [a_k, a_k + 1]$. The sequence $\{a_k\}$ characterizes the distribution of the oscillations of $f_\Omega(x)$. If $a_k = b + (k \ln^{1/q} k)^{p'}$, then $f_\Omega \in N_{pq+\varepsilon}(M)$ for each $\varepsilon > 0$, but $f_\Omega \notin N_{pq}(M)$. Let S be the set of all finite subsets from $\mathbb{Z}$. For a fixed $W \in S$ let $n_{pq}(W)$, $0 < p, q \le \infty$, be the set of sequences $a = \{a_m\}_{m \in \mathbb{Z}}$ endowed with the quasinorm

$$\|a\|_{n_{pq}(W)} = \left(\sum_{k=1}^\infty k^{q/p-1} \overline{a}_k(W)^q \right)^{1/q}$$

for $0 < p < \infty$, $0 < q < \infty$, and the quasinorm

$$\|a\|_{n_{p\infty}(W)} = \sup_{1 \le k < \infty} k^{1/p}\, \overline{a}_k(W)$$

for $q = \infty$, $0 < p \le \infty$, where

$$\overline{a}_k(W) \stackrel{\text{def}}{=} \sup_{\substack{e \in W \\ |e| > k}} \frac{1}{|e|} \left| \sum_{m \in e} a_m \right|.$$

Here $|e|$ is the number of elements in e.

Let $\{e^{2\pi imx}\}_{m\in\mathbb{Z}}$ be a trigonometric system, let $0<p<\infty$, $0<q\le\infty$, $\alpha\in\mathbb{R}$ and let W be a net in $\mathbb{Z}$.

Let $N_p^{\alpha q}(W)$ be the set of series $f=\sum_{m\in\mathbb{Z}} a_k e^{2\pi imx}$ such that

$$\|f\|_{N_p^{\alpha q}(W)}=\left(\sum_{k=1}^{\infty}\left(k^{\alpha+1}\sup_{\substack{|e|>k\\ e\in W}}\frac{1}{|e|}\Big\|\sum_{m\in e}a_m e^{2\pi imx}\Big\|_{L_p}\right)^q\right)^{1/q}<\infty$$

if $q<\infty$, or

$$\|f\|_{N_p^{\alpha\infty}(W)}=\sup_k k^{\alpha+1}\sup_{\substack{|e|>k\\ e\in W}}\frac{1}{|e|}\Big\|\sum_{m\in e}a_m e^{2\pi imx}\Big\|_{L_p}<\infty$$

if $q=\infty$.

We shall point out several properties of the spaces $N_{pq}(M)$, $n_{pq}(W)$ and $N_p^{\alpha q}(W)$.

1) If $1<p<\infty$, $1\le q\le\infty$, $M=\{e\subset\Omega:0<|e|<\infty\}$, $W=S$, then

$$N_{pq}(M)=L_{pq}(\Omega),\quad n_{pq}(W)=l_{pq},$$

where L_{pq} and l_{pq} are the classical Lorentz spaces.

2) If $1<p<\infty$, $\alpha>0$, $1\le q\le\infty$, $W_1=\{([2^{k-1},2^k]\cup[-2^{k-1},-2^k])\cap\mathbb{Z}:k\in\mathbb{N}\}$, then

$$N_p^{\alpha q}(W_1)=B_{pq}^{\alpha}[0,1],$$

where $B_{pq}^{\alpha}[0,1]$ is the Besov space.

3) For $0<q\le q_1\le\infty$ we have

$$N_{pq}(M)\hookrightarrow N_{pq_1}(M),\quad n_{pq}(W)\hookrightarrow n_{pq_1}(W),\quad N_p^{\alpha q}(W)\hookrightarrow N_p^{\alpha q_1}(W).$$

4) Let M be a net such that $\sup_{e\in M}|e|=\alpha<\infty$, let $0<p<p_1\le\infty$ and $0<q,q_1\le\infty$. Then

$$N_{p_1q_1}(M)\hookrightarrow N_{pq}(M),\quad n_{pq}(W)\hookrightarrow n_{p_1q_1}(W).$$

5) Let $0<p_0<p_1\le\infty$, $0<q\le\infty$, $0<\theta<1$ and let M be an arbitrary net in $\mathbb{R}^n$. If T is a semiadditive operator such that

$$T\colon A_0\to N_{p_0\infty}(M)\quad\text{with the norm } D_0,$$
$$T\colon A_1\to N_{p_1\infty}(M)\quad\text{with the norm } D_1,$$

then we also have $T\colon \overline{A}_{\theta q} \to N_{pq}(M)$ with the norm $\|T\| \le cD_0^{1-\theta}D_1^{\theta}$, where $1/p = (1-\theta)/p_0 + \theta/p_1$.

6) If W is an arbitrary net in $\mathbb{Z}$ and T is a semiadditive operator such that

$$T\colon A_0 \to n_{p_0\infty}(W) \quad \text{with the norm } D_0,$$
$$T\colon A_1 \to n_{p_1\infty}(W) \quad \text{with the norm } D_1,$$

then $T\colon \overline{A}_{\theta q} \to n_{pq}(W)$ and $\|T\| \le cD_0^{1-\theta}D_1^{\theta}$, where $1/p = (1-\theta)/p_0 + \theta/p_1$.

7) If W is an arbitrary net in $\mathbb{Z}$ and T is a semiadditive operator such that

$$T\colon A_0 \to N_p^{\alpha_0\infty}(W) \quad \text{with the norm } D_0,$$
$$T\colon A_1 \to N_p^{\alpha_1\infty}(W) \quad \text{with the norm } D_1,$$

then $T\colon \overline{A}_{\theta q} \to N_p^{\alpha q}(W)$ and $\|T\| \le cD_0^{1-\theta}D_1^{\theta}$, where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

2. Estimate of norm of integral convolution operator

Let

$$(Af)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,\mathrm{d}y = K * f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

be an integral convolution operator acting from L_p in L_q, where $L_p = L_p(\mathbb{R}^n)$ is the Lebesgue space.

For $1 \le p \le q \le \infty$ the Young inequality implies that $\|A\|_{L_p\to L_q} \le \|K\|_{L_r}$ where $1/r = 1 - 1/p + 1/q$. This estimate cannot be applied for operators with the power kernel $K(x) = |x|^{-\gamma}$. By the Hardy-Littlewood inequality the operator $Af = \int_{\mathbb{R}^n} f(y)|x-y|^{-\gamma}\,\mathrm{d}y$ is bounded if and only if $\gamma/n = 1 - 1/p + 1/q$.

Later on R. O'Neil in [5] proved the inequality

$$\|A\|_{L_p\to L_q} \le C\,\|K\|_{L_{r\infty}} \tag{2}$$

for $1 \le p < q \le \infty$, $1/r = 1 - 1/p + 1/q$, where $L_{r\infty}$ is the Marcinkiewicz-Lorentz space. This inequality is a refinement of the Young inequality and a generalization of the Hardy-Littlewood inequality.

Let us consider the problem of the lower estimate for the convolution operator (1). To solve the problem we shall rewrite the inequality (2) in the terms of net spaces in the following way:

$$\|A\|_{L_p\to L_q} \le C\,\|K\|_{N_{r\infty}(M)} \sim \sup_{e\in M} \frac{1}{|e|^{1/p-1/q}} \left|\int_e K(x)\,\mathrm{d}x\right|, \tag{3}$$

where M is the set of all bounded measurable subsets e of $\mathbb{R}^n$ and $|e|$ is the Lebesgue measure of e.

Thus, the new parameter M takes part in the inequality (3) and we can ask if there exists such net M_0 that the reverse inequality takes place.

Theorem 1. *Assume that $1 < p \le q < +\infty$ and $K(x) \ge 0$, $x \in \mathbb{R}^n$. If the operator (1) is bounded from L_p in L_q, then there exists a constant $C = C(p, q, n)$ such that*

$$C \sup_{e \in Q(C)} \frac{1}{|e|^{1/p-1/q}} \int_e K(x)\,\mathrm{d}x \le \|A\|_{L_p \to L_q},$$

where $Q(C)$ is the class of all sets e of finite measure such that $|e+e| \le C|e|$.

P r o o f. Let e be an arbitrary element of $Q(C)$.

$$\begin{aligned}\|A\|_{L_p \to L_q} &= \sup_{\substack{\|f\|_p=1\\ \|g\|_{q'}=1}} \left| \int_{R^n} \int_{R^n} K(x-y) f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y \right| \\ &\ge \frac{1}{|e+e|^{1/p} |e|^{1/q'}} \left| \int_e \int_{e+e} K(x-y)\,\mathrm{d}x\,\mathrm{d}y \right|.\end{aligned}$$

Since $K(x) \ge 0$ and $e \subset [(e+e) - y]$ for every $y \in e$, we have

$$\begin{aligned}\|A\|_{L_p \to L_q} &\ge \frac{1}{|e+e|^{1/p} |e|^{1/q'}} \int_e \int_e K(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{|e+e|^{1/p} |e|^{-1/q}} \int_e K(x)\,\mathrm{d}x \\ &\ge C^{-1/p} \frac{1}{|e|^{1/p-1/q}} \int_e K(x)\,\mathrm{d}x.\end{aligned}$$

□

Theorem 2. *Assume that $1 < p < q < +\infty$. If the operator (1) is bounded from L_p in L_q, then there exists a constant $C = C(p, q, n)$ such that*

$$C \sup_{e \in M_0} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)\,\mathrm{d}x \right| \le \|A\|_{L_p \to L_q},$$

where M_0 is the set of all n-dimensional parallelepipeds in $\mathbb{R}^n$.

3. Fourier transform multipliers

Let F and F^{-1} be the direct and inverse Fourier transforms in $\mathbb{R}$, respectively. A function u is called the Fourier transform multiplier from the Lebesgue function space $L_p(\mathbb{R})$ into the Lebesgue space $L_q(\mathbb{R})$ if the operator $T_u(f) = F^{-1}uFf\colon L_p(\mathbb{R}) \to L_q(\mathbb{R})$ is bounded. Denote by $M(L_p \to L_q)$ the set of all multipliers from L_p in L_q. This set is a linear normed space with the norm

$$\|u\|_{M(L_p\to L_q)} = \sup_{\|f\|_{L_p}\neq 0} \frac{\|T_u(f)\|_{L_q}}{\|f\|_{L_p}}.$$

In [4] the following theorem was proved.

Hörmander's theorem. *If* $1 < p < 2 \le q \le \infty$ *and* $1/r = 1/p - 1/q$, *then* $L_{r\infty} \hookrightarrow M(L_p \to L_q)$, *i.e.*,

$$\|u\|_{M(L_p\to L_q)} \le C\,\|u\|_{L_{r\infty}}.$$

As in the case of the convolution operator we shall rewrite this inequality in the terms of net spaces:

$$\|u\|_{M(L_p\to L_q)} \le C \sup_{e\in E} \frac{1}{|e|^{1/p'+1/q}} \left| \int_e u(x)\,\mathrm{d}x \right|.$$

Here E is the set of all compact sets e with positive measure $|e| > 0$.

In a similarity to the previous section we shall deal with the problem of finding a net $E_0 \subset E$ such that the inequality

$$\sup_{e\in E_0} \frac{1}{|e|^{1-1/r}} \left| \int_e u(x)\,\mathrm{d}x \right| \le \|u\|_{M(L_p\to L_q)}$$

holds.

A set of the form $\Omega = \bigcup_{0\le k\le N}([a,b]+kh)$, $h > b-a$, will be called a harmonic interval in $\mathbb{R}$. It is obvious that any interval in $\mathbb{R}$ is a harmonic interval.

Theorem 3. *Suppose that* $1 < p < 2 \le q \le \infty$, $1/r = 1/p - 1/q$ *and* M_0 *is the set of all intervals of positive measure in* $\mathbb{R}$. *Then for any non-negative function* $u \in M(L_p \to L_q)$ *the following inequality holds:*

$$\sup_{e\in M_0} \frac{1}{|e|^{1-1/r}} \int_e u(x)\,\mathrm{d}x \le C\,\|u\|_{M(L_p\to L_q)}. \tag{4}$$

P r o o f. Let $[a,b]$ be an arbitrary segment, $\delta = \pi/(b-a) > 0$ and let $u \in M(L_p \to L_q)$, $u \neq 0$.

By definition of the class $M(L_p \to L_q)$, we have

$$\begin{aligned}
\|u\|_{M(L_p\to L_q)} &= \sup_{\|f\|_p=1} \|F^{-1}uFf\|_{L_q} \\
&= \sup_{\substack{\|f\|_p=1\\ \|g\|_{q'}=1}} \int_{-\infty}^{+\infty} (F^{-1}uFf)(y)\overline{g}(y)\,\mathrm{d}y \\
&\geq \sup_{\substack{|e|>0\\ |w|>0}} \frac{1}{|e|^{1/q'}}\frac{1}{|w|^{1/p}} \int_{-\infty}^{+\infty} u(x) \int_e \mathrm{e}^{ixy}\,\mathrm{d}y \int_w \mathrm{e}^{ixz}\,\mathrm{d}z\,\mathrm{d}x \\
&\geq \frac{1}{\delta^{1/q'+1/p}} \int_{-\infty}^{+\infty} u(x+a)\left|\int_0^{\delta} \mathrm{e}^{ixy}\,\mathrm{d}y\right|^2 \mathrm{d}x \\
&\geq \frac{1}{\delta^{1/q'+1/p}} \int_{-\infty}^{+\infty} u(x+a)\left(\frac{\sin\frac{\delta x}{2}}{\frac{x}{2}}\right)^2 \mathrm{d}x.
\end{aligned}$$

Since $u \geq 0$, $u \neq 0$, we obtain

$$\begin{aligned}
\|u\|_{M(L_p\to L_q)} &\geq \frac{\delta^2}{\delta^{1/q'+1/p}} \int_0^{\pi/\delta} u(x+a)\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)^2 \mathrm{d}x \\
&\geq \left(\frac{2}{\pi}\right)^2 \delta^{1/p'+1/q} \int_0^{\pi/\delta} u(x+a)\,\mathrm{d}x \\
&= \left(\frac{2}{\pi}\right)^2 \pi^{1/p'+1/q} \frac{1}{b-a} \int_a^b u(x)\,\mathrm{d}x.
\end{aligned}$$

Since $[a,b]$ was arbitrary, inequality (4) holds. □

4. Inequalities of Hardy-Littlewood type

The following theorem of Hardy and Littlewood is well known: Let $p \geq 2$ and let $a = \{a_m\}_{m\in\mathbb{Z}}$ be a sequence of integers such that for its non-increasing rearrangement $a^* = \{a_k^*\}_{k=1}^{\infty}$ the series

$$\sum_{k=1}^{\infty} k^{p-2} a_k^{*\,p}, \tag{5}$$

is convergent. Then the series $\sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{2\pi imx}$ converges to a function f in the space $L_p[0,1]$ and

$$\|f\|_{L_p}^p \leq C \sum_{k=1}^{\infty} k^{p-2} a_k^{*\,p} = C\,\|a\|_{l_{p'p}}^p. \tag{6}$$

Moreover, if $p \geq 2$ and f belongs to the Lorentz space $L_{p'p}[0,1]$, then the sequence of its Fourier coefficients belongs to l_p and

$$\sum_{k\in\mathbb{Z}} |a_k|^p \leq \int_0^{2\pi} t^{p-2} f^{*\,p}(t)\,\mathrm{d}t, \tag{7}$$

where f^* is the non-increasing rearrangement of f.

In a certain sense this result is a counterpart to the Parseval identity in L_p. For instance, if one considers functions with monotone [1] or quasi-monotone coefficients [3], then the reversed inequality (6) also holds. Thus, if f is a function with monotone or quasi-monotone coefficients, then the convergence of (5) is equivalent to the inclusion $f \in L_p$.

Incidentally, it is not true in general that if a function f belongs to L_p for some $p > 2$, then (5) converges. Hence there arises the question of necessary conditions for f to belong to L_p, that is, the problem of lower estimates of $\|f\|_{L_p}$ in terms of series of the form (5).

This section is devoted to this problem and to a similar one concerning the reverse of (7). The idea of the proof is to get weak estimates in the terms of net spaces; it is then applied to the interpolation Theorem 2.

Let $f \in L_1[0,1]$ and let

$$a_k = \int_0^1 f(x)\mathrm{e}^{2\pi ikx}\,\mathrm{d}x, \quad k \in \mathbb{Z},$$

be correponding Fourier coefficients with respect to the trigonometric system $\{\mathrm{e}^{2\pi ikx}\}_{k\in\mathbb{Z}}$.

Theorem 4. *Assume that* $2 < p < \infty$, M_0 *is the set of all closed intervals of* $[0,1]$, M_1 *is the set of all compact subsets of* $[0,1]$ *and* $f \sim \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{2\pi ikx}$. *Then*

$$c_0\|f\|_{N_{p'q}(M_0)} \leq \|a\|_{l_{pq}} \leq c_1\|f\|_{N_{p'q}(M_1)}. \tag{8}$$

In particular,

$$c_0 \int_0^\infty t^{p-2}\overline{f}^p(t, M_0)\,\mathrm{d}t \leq \sum_{m\in\mathbb{Z}^n} |a_k|^p \leq c_1 \int_0^\infty t^{p-2}\overline{f}^p(t, M_1)\,\mathrm{d}t,$$

where c_1 *and* c_2 *are constants depending only on the parameter* p *and on the dimension* n.

P r o o f. Using the fact that non-increasing permutations of the sequence $\{|e|^{-1/p}\min(|e|,1/k)\}_{k=1}^{\infty}$ can be estimated by $\{k^{-1/p'}\}_{k=1}^{\infty}$, we obtain that

$$\frac{1}{|e|^{1/p}}\left|\int_e f(x)\,\mathrm{d}x\right| \le C_1\sum_{k=-\infty}^{\infty} a_k\frac{\min(|e|,1/k)}{|e|^{1/p}} \le C_2\sum_{k=1}^{\infty} a_k^* k^{-1/p'} = C_2\|a\|_{l_{p1}}$$

for any $e\in M_0$. Hence

$$\|f\|_{N_{p'\infty}} \le C\,\|a\|_{l_{p1}}.$$

Since the value of p is an arbitrary number such that $1<p<\infty$, interpolation properties of the space N_{pq} imply the first inequality in (8). The second one follows from the Marcinkiewicz-Zygmund inequality and from the equality $N_{pq}(M_1)=L_{pq}[0,1]$ (see [3]). □

We say that f is monotone in the generalized sense if there is a number $c>0$ such that for every $x\in(0,1]$

$$|f(x)| \le c\,\frac{1}{x}\left|\int_0^x f(t)\,\mathrm{d}t\right|.$$

Corollary 1. *Let $1<p<2$ and let f be monotone in the generalized sense. Then $f\in L_{pq}[0,1]$ if and only if $\{a_k\}\in l_{p'q}$.*

Theorem 5. *Let $p>2$. Let M_1 be the set of all finite subsets in $\mathbb{Z}$, let M_0 be the set of all arithmetic progressions in $\mathbb{Z}$ and let $f=\sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{2\pi ikx}$. Then*

$$c_0\|a\|_{n_{p'q}(M_0)} \le \|f\|_{L_{pq}} \le c_1\|a\|_{n_{p'q}(M_1)}.$$

We say that a sequence of complex numbers $\{a_m\}_{m\in\mathbb{Z}}$ is monotone in the generalized sense if

$$|a_k| \le c\,\frac{1}{|k|}\left|\sum_{m=-k}^{k} a_m\right|, \quad k\in\mathbb{N}.$$

Let λ be the set of functions f whose Fourier coefficients are monotone in the generalized sense. The set λ contains functions with monotone (see [5]) or quasimonotone coefficients (see [2]).

Corollary 2. *Let $2<p<\infty$ and $f\in\lambda$. Then $f\in L_{pq}[0,1]$ if and only if $\{a_k\}\in l_{p'q}$.*

Corollary 3. *Let $1 < p < 2$ and let $\{a_m\}_{m=1}^{\infty}$ be a sequence of complex numbers such that*

$$B = \left(\sum_{k=0}^{\infty} \left(\sum_{2^k \le |m| < 2^{k+1}} m^{r/p'-1} |m\Delta a_m|^r \right)^{p/r} \right)^{1/p} + \sup_{m>0} m^{1/p'} |a_m| < \infty$$

for some $1 \le r \le \infty$, where $\Delta a_m = a_m - a_{m+1}$.

Then $\sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{2\pi imx}$ is the Fourier series of some function $f \in L_p$ and

$$\|f\|_{L_p} \le cB,$$

where c depends only on the parameters p, r.

5. Nikol'skij's inequality for different metrics

In the theory of spaces of differentiable functions, a fundamental role is played by Nikol'skij's inequality for different metrics [5]. This inequality and the Riesz theorem imply the following result: Let $1 \le p < r \le \infty$, $f \in L_p[0,1]$, $f \sim \sum_{m\in\mathbb{Z}} a_m \mathrm{e}^{2\pi imx}$, $S_k(f) = \sum_{m=-k}^{k} a_m \mathrm{e}^{2\pi imx}$, $k \in \mathbb{N}$. Then

$$\sup_{k\ge 1} k^{1/r-1/p} \|S_k(f)\|_{L_r[0,1]} \le c\|f\|_{L_p[0,1]}. \tag{9}$$

Thus, for a function $f \in L_p[0,1]$, the sequence $\{\|S_k(f)\|_{L_r[0,1]}\}$ of L_r-norms of its partial sums does not increase faster than $O(k^{1/r-1/p})$. It gives rise to the question whether the inequality (9) characterizes the increase of L_r-norm of partial sums of functions from L_p.

We shall prove a statement which refines the inequality (9).

Theorem 6. *Let $1 < p < r \le \infty$, $\alpha = 1/r - 1/p$ and $0 < q \le \infty$. For $f \in L_{pq}[0,1]$, $f \sim \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{2\pi ikx}$, set $S_m(f) = \sum_{k=-m}^{m} a_k \mathrm{e}^{2\pi ikx}$, $m \in \mathbb{N}$.*

Then

$$L_{pq}[0,1] \hookrightarrow N_p^{\alpha q}(W_0),$$

where W_0 consists of all segments in $\mathbb{Z}$, and

$$\left(\sum_{k=0}^{\infty} \left(k^{1/h} \sup_{m\ge k} \frac{1}{m} \|S_m(f)\|_{L_r[0,1]} \right)^q \frac{1}{k} \right)^{1/q} \le c\,\|f\|_{L_{pq}[0,1]}.$$

P r o o f. Nikol'skij's inequality and the Riesz theorem imply

$$\|f\|_{N_r^{\alpha\infty}(W_0)} \le C\,\|f\|_{L_p},$$

that is, for the embedding operator we have

$$T\colon L_p[0,1] \to N_r^{\alpha\infty}(W_0).$$

The interpolation properties of the spaces $N_p^{\alpha q}(W)$ imply

$$T\colon L_{pq}[0,1] \to N_r^{\alpha q}(W_0)$$

or

$$L_{pq}[0,1] \hookrightarrow N_p^{\alpha q}(W_0).$$

□

Corollary 4. *Let $1 < p < r \le \infty$, $0 < q \le \infty$, and let $f \in L_{pq}[0,1]$, $f \sim \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{2\pi ikx}$. Then*

$$\|S_m(f)\|_{L_r[0,1]} = o(m^{1/p-1/r})$$

and

$$\left(\sum_{k=1}^{\infty} \frac{\left(k^{1/r-1/p}\|S_k(f)\|_{L_r[0,1]}\right)^q}{k}\right)^{1/q} \le c\,\|f\|_{L_{pq}[0,1]},$$

where $S_m(f) = \sum_{k=-m}^{m} a_k \mathrm{e}^{2\pi ikx}$, $m \in \mathbb{N}$. In particular,

$$\left(\sum_{k=1}^{\infty} \frac{k^{p/r-1}\|S_k(f)\|_{L_r[0,1]}^p}{k}\right)^{1/p} \le c\,\|f\|_{L_p[0,1]}.$$

Theorem 7. *Let $1 < p < r \le \infty$ and $0 < q < \infty$. Then there exists a function $f \in L_{pq}[0,1]$, $f \sim \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{2\pi ikx}$, such that for every $\varepsilon > 0$*

$$\sum_{k=1}^{\infty} \frac{\left(k^{1/r-1/p}\|S_k(f)\|_{L_r[0,1]}\right)^{q-\varepsilon}}{k} = \infty,$$

where $S_m(f) = \sum_{k=-m}^{m} a_k \mathrm{e}^{2\pi ikx}$, $m \in \mathbb{N}$.

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