NORM CONVOLUTION INEQUALITIES IN LEBESGUE SPACES

E. NURSULTANOV, S. TIKHONOV, AND N. TLEUKHANOVA

Abstract. We obtain upper and lower estimates of the (p, q) norm of the convolution operator. The upper estimate sharpens the Young-type inequalities due to O'Neil and Stepanov.

1. Introduction

Let $1 \leqslant p \leqslant \infty$, $L_p \equiv L_p(\mathbb{R})$ and let the convolution operator be given by

$$(1.1) \qquad (Af)(x) = (K * f)(x) = \int\limits_{\mathbb{R}} K(x-y)f(y)dy.$$

The Young convolution inequality

$$\|A\|_{L_p \to L_q} \leqslant \|K\|_{L_r}, \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \qquad 1 \leqslant p \leqslant q \leqslant \infty,$$

plays a very important role both in Harmonic Analysis and PDE theory. We note however that this estimate does not allow us to deal with power kernels such as $K(x) = |x|^{-\gamma}$, $\gamma > 0$.

Young's estimates were generalized by O'Neil [ON] who showed that for $1 < p < q < \infty$ and $1/r = 1 - 1/p + 1/q$

$$(1.2) \qquad \|A\|_{L_p \to L_q} \leqslant C \, \|K\|_{L_{r,\infty}} := C \sup_{t>0} \; t^{1/r} K^*(t),$$

where $K^*(t) = \inf\big\{\sigma : \mu\{x \in \Omega : |f(x)| > \sigma\} \leqslant t\big\}$ is the decreasing rearrangement of K. In particular, this gives the Hardy-Littlewood fractional integration theorem, which corresponds to the model case of convolution by $K(x) = |x|^{-1/r}$.

Another extension of Young's inequality was proved by Stepanov [Stp] using the Wiener amalgam space $W\big(L_{r,\infty}[0,1],\, l_{r,\infty}(\mathbb{Z})\big)$ (see e.g. [Fe]): for $1 < p < q < +\infty$ and $1/r = 1 - 1/p + 1/q$ one has

$$(1.3) \qquad \|A\|_{L_p \to L_q} \leqslant C \, \|K\|_{W(L_{r,\infty}[0,1],\, l_{r,\infty}(\mathbb{Z}))},$$

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where

$$\|K\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))} := \Big\| \|\tilde{K}\|_{L_{r,\infty}[0,1]} \Big\|_{l_{r,\infty}(\mathbb{Z})} := \sup_{n\in\mathbb{N}} \; n^{1/r} \left(\sup_{0\leqslant t\leqslant 1} \; t^{1/r}\tilde{K}^*(t,\cdot) \right)_n^*,$$

and $\tilde{K}(x,m) := K(m+x), \quad m\in\mathbb{Z},\ x\in[0,1]$. In [Stp] it was also shown that inequalities (1.2) and (1.3) are not comparable.

The aim of the present paper is to give upper and lower estimates of $\|A\|_{L_p\to L_q}$ so that the upper estimate improves both (1.2) and (1.3). To formulate our main results, we will need the following definitions.

Let I be an interval with $|I| = d$. Then $T_I = \{I + kd\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}$, i.e., $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}(I+kd)$. We define two collections of sets $\mathfrak{L}(I) \subset \mathfrak{U}(I)$:

$$\mathfrak{L}(I) = \left\{ e \;:\; e = \bigcup_{k=1}^{m} \big([a,b]+kd\big), \quad [a,b]\subseteq I, \quad m\in\mathbb{N} \right\} \tag{1.4}$$

and

$$\mathfrak{U}(I) = \left\{ e \;:\; e = \bigcup_{k=1}^{m} \omega_k, \quad m\in\mathbb{N} \right\},$$

where $\{\omega_k\}_1^m$ is any collection of compact sets of equal measure $|\omega_k| \leqslant d$ and such that each ω_k belongs to a different elements of T_I.

Theorem. *Let $1<p<q<\infty$ and $K\in L_{loc}$. Then for $Af = K*f$ we have*

$$C_1 \sup_I \sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)dx \right| \leqslant \|A\|_{L_p\to L_q} \leqslant C_2 \inf_I \sup_{e\in\mathfrak{U}(I)} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)dx \right|, \tag{1.5}$$

where the constants C_1 and C_2 depend only on p and q.

For the certain regular kernels K, for instance, monotone or quasi-monotone, the upper and lower bounds in (1.5) coincide, that is, we get the equivalent relation for $\|A\|_{L_p\to L_q}$. More precisely, we call a locally integrable function $K(x)$ *weakly monotone* if there exists a constant $C>0$ such that for any $x\in\mathbb{R}\setminus\{0\}$

$$|K(x)| \leqslant C\left| \frac{1}{x}\int_0^x K(t)dt \right|.$$

Corollary. *Let $1<p<q<\infty$ and $K\in L_{loc}$ be a weakly monotone function. Then a necessary and sufficient condition for $Af = K*f$ to be bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ is*

$$\sup_{|x|>0} \frac{1}{|x|^{1/p-1/q}} \left| \int_0^x K(y)dy \right| < \infty.$$

Moreover,

$$\|A\|_{L_p\to L_q} \approx \sup_{|x|>0} \frac{1}{|x|^{1/p-1/q}} \left| \int_0^x K(y)dy \right|.$$

By C, C_i, c we will denote positive constants that may be different on different occasions. We write $F \approx G$ if $F \leqslant C_1 G$ and $G \leqslant C_2 F$ for some positive constants C_1 and C_2 independent of essential quantities involved in the expressions F and G.

The paper is organized as follows. In section 2 we obtain a required version of the Riesz Lemma for rearrangements (see, e.g., [St]). Section 3 and 4 are devoted to the estimates of $\|A\|_{L_p\to L_q}$ from above and below, correspondingly. We conclude with Section 5, where we show that the right-hand side estimate in (1.5) implies both (1.2) and (1.3) but the reverse does not hold in general.

2. Rearrangement inequalities

First, we denote the decreasing rearrangement of f on $\mathbb{Z}^n$ by f^*. We also denote $f^{**}(n) := \frac{1}{n}\sum_{k=1}^n f^*(k)$.

Lemma 2.1. *Let functions $f, g,$ and K are defined on $\mathbb{Z}^n$; then*

$$\sum_{k\in\mathbb{Z}} g(k)(K*f)(k) \leqslant 2\sum_{r=1}^{\infty} r g^{**}(r) f^{**}(r) K^{**}(r). \tag{2.1}$$

Proof. From $f^{**}(n) = \sup\limits_{\substack{|e|=n\\ e\subset\mathbb{Z}}} \frac{1}{|e|}\sum\limits_{s\in e}|f(s)|$ (see [BS, Ch. 2, §3]) and the Hardy-Littlewood inequality [BS, p.44], we write

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} g(k)(K*f)(k) &\leqslant \sum_{r=1}^{\infty} g^*(r)(K*f)^{**}(r)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m\in\mathbb{Z}} |f(m)| \frac{1}{|e|}\sum_{s\in e}|K(s-m)|\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m=1}^{\infty} f^*(m) \left(\frac{1}{|e|}\sum_{s\in e}|K(s-\cdot)|\right)^{**}(m)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m=1}^{\infty} f^*(m) \left(\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\right)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sum_{m=1}^{\infty} f^*(m) \left(\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\right).
\end{aligned}$$

We consider

$$\Phi(r,m)=\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}}\frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\,.$$

If $r\leqslant m$, then

$$\Phi(r,m)\leqslant\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sum_{s\in e}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}}\frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}|K(s-t)|=K^{**}(m)$$

and if $m\leqslant r$, then

$$\Phi(r,m)\leqslant\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}}\frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sum_{s\in e}|K(s-t)|=K^{**}(r).$$

Hence, we get

$$\Phi(r,m)\leqslant K^{**}(\max\{r,m\}).$$

Therefore,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}^{\infty}g(k)(K*f)(k)&\leqslant\sum_{r=1}^{\infty}g^*(r)\sum_{m=1}^{\infty}f^*(m)K^{**}(\max\{r,m\})\\&=\sum_{r=1}^{\infty}g^*(r)K^{**}(r)\sum_{m=1}^{r}f(m)^*\\&\quad+\sum_{r=1}^{\infty}g^*(r)\sum_{m=r+1}^{\infty}f^*(m)K^{**}(m)\\&=\sum_{r=1}^{\infty}rg^*(r)K^{**}(r)f^{**}(r)+\sum_{m=1}^{\infty}f^*(m)K^{**}(m)\sum_{r=1}^{m}g^*(r)\\&\leqslant 2\sum_{r=1}^{\infty}rg^{**}(r)K^{**}(r)f^{**}(r).\end{aligned}$$

The proof is complete. □

The continuous analogue of the previous lemma is the following result.

Lemma 2.2. *Let f and g be measurable functions on $[0,d]$ and K be measurable on $[-d,d]$. Then*

$$\int_0^d g(y)\int_0^d f(x)K(y-x)\,dx\,dy \qquad\qquad (2.2)$$

$$\leqslant 2\int_0^d tg^{**}(t)f^{**}(t)\left(\sup_{\substack{e\subset[-d,d]\\ |e|=t}}\frac{1}{|e|}\int_e|K(x)|\,dx\right)dt.$$

Proof. Similarly to the proof of Lemma 2.1, we have

$$\int_0^d g(y)\,(K*f)\,(y)dy \leqslant \int_0^d g^*(s)\int_0^d f^*(t)\sup_{\substack{e\subset[0,d]\\|e|=s}}\sup_{\substack{\omega\subset[0,d]\\|\omega|=t}}\frac{1}{|e|}\frac{1}{|\omega|}\int_e\int_\omega |K(y-x)|\,dxdy$$
$$=\int_0^d g^*(s)\int_0^d f^*(t)\Phi(s,t)dtds.$$

Further, for $s\leqslant t$, we get

$$\Phi(s,t)\leqslant \sup_{\substack{e\subset[0,d)\\|e|=s}}\frac{1}{|e|}\int_e \sup_{\substack{\omega\subset[0,d]\\|\omega|=t}}\frac{1}{|\omega|}\int_\omega |K(y-x)|\,dxdy \leqslant \sup_{\substack{\omega\subset[-d,d]\\|\omega|=t}}\frac{1}{|\omega|}\int_\omega |K(x)|\,dx,$$

and for $s\geqslant t$,

$$\Phi(s,t)\leqslant \sup_{\substack{e\subset[-d,d]\\|e|=s}}\frac{1}{|e|}\int_e |K(y)|\,dy.$$

Finally, as in the proof of Lemma 2.1, we have

$$\int_0^d g(y)\,(K*f)\,(y)dy\leqslant 2\int_0^d tg^{**}(t)f^{**}(t)\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\int_e |K(x)|\,dx. \qquad \square$$

3. Proof of upper bound for $\|A\|_{L_p\to L_q}$

Let $d>0$, $I=[0,d)$, and $T_I=\{(md,(m+1)d]\}_{m\in\mathbb{Z}}$ be the corresponding partition of $\mathbb{R}$. For a locally integrable function $K(x)$ we put $K(x)=K_1(x,d)+K_2(x,d)$, where

$$K_1(x,d)=\begin{cases} K(x), & \text{if } \ x\in(2md,(2m+1)d]\,, \quad m\in\mathbb{Z}\\ 0, & \text{if } \ x\in((2m-1)d,2md]\,, \quad m\in\mathbb{Z}\end{cases}$$

and

$$K_2(x,d)=\begin{cases} 0, & \text{if } \ x\in(2md,(2m+1)d]\,, \quad m\in\mathbb{Z}\\ K(x), & \text{if } \ x\in((2m-1)d,2md]\,, \quad m\in\mathbb{Z}.\end{cases}$$

Then we write the convolution operator $Af=f*K$ as $A=A_1+A_2$, where $A_if=f*K_i,\ \ i=1,2$, we have

$$\|A\|_{L_p\to L_q}\leqslant 2\max_{i=1,2}\|A_i\|_{L_p\to L_q}\,. \tag{3.1}$$

Let $d>0$ for $k\in\mathbb{Z}$ and $x\in[0,d]$, we denote

$$\widetilde{f}(x,k):=f(x+kd),$$
$$\widetilde{g}(x,k):=g(x+kd),$$
$$\widetilde{K}_i(x,k):=K_i(x+kd).$$

We are going to estimate the following quantity

$$J_i:=\int_{\mathbb{R}}g(y)\int_{\mathbb{R}}f(x)K_i(y-x)\,dxdy,\quad i=1,2.$$

Let us write it as follows

$$J_i=\sum_{k\in\mathbb{Z}}\int_0^d g(y+kd)\sum_{m\in\mathbb{Z}}\int_0^d f(x+md)K_i\Big((y-x)+(k-m)d\Big)\,dxdy$$

$$\equiv\sum_{k\in\mathbb{Z}}\int_0^d\tilde{g}(y,k)\sum_{m\in\mathbb{Z}}\int_0^d\tilde{f}(x,m)\tilde{K}_i(y-x,k-m)\,dxdy. \tag{3.2}$$

To estimate this functional, we first use Lemma 2.2:

$$J_i\leqslant 2\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_0^d t\tilde{f}^{(**)_1}(t,m)\tilde{g}^{(**)_1}(t,k)\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\left|\int_e\tilde{K}_i(x,k-m)dx\right|dt$$

$$=2\int_0^d t\left(\sum_{k\in\mathbb{Z}}\tilde{g}^{(**)_1}(t,k)\sum_{m\in\mathbb{Z}}\tilde{f}^{(**)_1}(t,m)\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_i(x,k-m)\right|dx\right)dt,$$

where

$$\tilde{f}^{(**)_1}(t,m)=\frac{1}{t}\int_0^t\tilde{f}^{*_1}(t,m)dt,\quad m\in\mathbb{Z},$$

$$\tilde{g}^{(**)_1}(t,k)=\frac{1}{t}\int_0^t\tilde{g}^{*_1}(t,k)dt,\quad k\in\mathbb{Z},$$

and $\tilde{f}^{*_1}(t,m),\tilde{g}^{*_1}(t,k)$ are decreasing rearrangements of $\tilde{f}(x,m),\tilde{g}(x,k)$ with respect to x and with fixed m and k, correspondingly.

Applying now Lemma 2.1, we get

$$J_i\leqslant 4\int_0^d t\sum_{s=1}^{\infty}s\tilde{f}^{**}(t,s)\tilde{g}^{**}(t,s)\left(\sup_{\substack{|\omega|=s\\\omega\subset\mathbb{Z}}}\frac{1}{|\omega|}\sum_{m\in e}\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_i(x,m)\right|dx\right)dt$$

$$\equiv 4\int_0^d t\sum_{s=1}^{+\infty}s\tilde{f}^{**}(t,s)\tilde{g}^{**}(t,s)\,F_d(t,s;K_i)\,dt,$$

where

$$\tilde{f}^{**}(t,s) = \frac{1}{s}\sum_{l=1}^{s}\left(\tilde{f}^{(**)_1}(t,\cdot)\right)_l^{*_2},$$

$$\tilde{g}^{**}(t,s) = \frac{1}{s}\sum_{l=1}^{s}\left(\tilde{g}^{(**)_1}(t,\cdot)\right)_l^{*_2}.$$

Then writing

$$(ts)\tilde{g}^{**}(t,s)\tilde{f}^{**}(t,s)F_d(t,s;K_i) \leqslant \left((ts)^{\frac{1}{p}-\frac{1}{q}}\tilde{f}^{**}(t,s)\right)\left(\tilde{g}^{**}(t,s)\right)\left(\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{\frac{1}{r}}F_d(t,s;K_i)\right)$$

and using Hölder's inequality with parameters q and q' and the fact that $L_{pq}\hookrightarrow L_{pq_1}$ for $q\leqslant q_1$, we get

$$\int_0^d\sum_{s=1}^{\infty} ts\tilde{f}^{**}(t,s)\tilde{g}^{**}(t,s)F_d(t,s;K_i)dt \leqslant 4\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_i)\left(\sum_{s\in\mathbb{N}}\int_0^d\left(\tilde{g}^{**}(t,s)\right)^{q'}dt\right)^{1/q'} \left(\sum_{s\in\mathbb{N}}\int_0^d\left(\tilde{f}^{**}(t,s)\right)^{p}dt\right)^{1/p}.$$

Then by Hardy's inequality,

$$\left|\int_{\mathbb{R}} g(y)\int_{\mathbb{R}} f(x)K_i(y-x)dxdy\right| \leqslant C\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_i)\left\|f\right\|_{L_p}\left\|g\right\|_{L_{q'}}.$$

Thus,

$$\left\|A_i\right\|_{L_p\to L_q} \leqslant C\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_i),\quad i=1,2. \tag{3.3}$$

Note that by definition, K_1 and K_2 satisfy

$$\operatorname{supp}\tilde{K}_1(x,s)\subset[-d,0]\times\mathbb{Z},\qquad \operatorname{supp}\tilde{K}_2(x,s)\subset[0,d]\times\mathbb{Z}.$$

Therefore,

$$\sup_{\substack{e\subset[-d,d]\\ |e|=t}}\frac{1}{|e|}\int_e\left|K_1(x,k)\right|dx = \sup_{\substack{e\subset[-d,0]\\ |e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_1(x,k)\right|dx,\quad k\in\mathbb{Z}$$

and

$$\sup_{\substack{e\subset[-d,d]\\ |e|=t}}\frac{1}{|e|}\int_e\left|K_2(x,k)\right|dx = \sup_{\substack{e\subset[0,d]\\ |e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_2(x,k)\right|dx,\quad k\in\mathbb{Z}.$$

Then

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_1)=\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}\sup_{|\omega|=s}\frac{1}{(ts)^{\frac{1}{p}-\frac{1}{q}}}\sum_{m\in\omega}\sup_{\substack{|e|=t\\ e\subset[-d,0]}}\int_e\left|\tilde{K}_1(x,m)\right|dx.$$

For any $m\in\mathbb{Z}$ and $t\in(0,d]$ we find $e_{m,t}\subset[-d,0]$ such that $|e_{m,t}|=t$ and

$$\begin{aligned}\sup_{|e|=t}\int_e\left|\tilde{K}_1(x,m)\right|dx&\leqslant 2\int_{e_{m,t}}\left|\tilde{K}_1(x,m)\right|dx\\ &=2\int_{e_{m,t}}|K_1(x+md)|\,dx=2\int_{e_{m,t}+md}|K(x)|\,dx.\end{aligned}$$

The set $\eta_m=e_{m,t}+md$ of measure t for different m belongs to different elements of $T_d=\{nd,(n+1)d\}_{n\in\mathbb{Z}}$. So, for $0<t\leqslant d$ and $r\in\mathbb{N}$ we have

$$\sup_{|\omega|=s}\frac{1}{(ts)^{\frac{1}{p}-\frac{1}{q}}}\sum_{m\in\omega}\sup_{|e|=t}\int_e\left|\tilde{K}_1(x,m)\right|dx\leqslant 2\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx.$$

Therefore, we obtain

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_1)\leqslant 2\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx$$

and, similarly,

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_2)\leqslant 2\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx.$$

Combining this with (3.1) and (3.3), we get

$$\|A\|_{L_p\to L_q}\leqslant C\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx$$

and using an arbitrary choice of $d>0$,

$$\|A\|_{L_p\to L_q}\leqslant C\sup_{d>0}\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx$$

with a constant C depending on p and q. Since the norms of operators $Af=K*f$ and $A_tf=K_t*f$, where $K_t(x)=K(x+t)$, $t\geqslant 0$ coincide, the last estimate implies

$$\|A\|_{L_p\to L_q}\leqslant C\sup_I\sup_{e\in\mathfrak{U}(I)}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx.$$

To finish this proof, it is sufficient to show the following

Lemma 3.1. *Let $0<\gamma\leqslant 1$ and K be locally integrable. Then for any $e\in\mathfrak{U}(I)$ there exists $e'\in\mathfrak{U}(I)$ such that*

$$\frac{1}{|e|^{\gamma}}\int_e|K(x)|dx\leqslant 2^{3-\gamma}\frac{1}{|e'|^{\gamma}}\left|\int_{e'}K(x)dx\right|.$$

Proof. Since $e \in \mathfrak{U}(I)$ we have $e = \bigcup_{k=1}^{m} \omega_k$, where $|\omega_k| = \omega < d, k = \overline{1,m}$ and ω_k belong to different elements of $T_I = \{I + kd\}_{r\in\mathbb{Z}}$.

For any w_k let us define

$$\omega_k^1 := \Big\{x \in w_k : K(x) \geqslant 0\Big\} \qquad \text{and} \qquad \omega_k^2 = \Big\{x \in \omega_k : K(x) < 0\Big\}.$$

Then

$$\int\limits_{\omega_k} |K(x)|dx = \int\limits_{\omega_k^1} K(x)dx - \int\limits_{\omega_k^2} K(x)dx \leqslant 2\max\left\{\Big|\int\limits_{\omega_k^1} K(x)dx\Big|, \Big|\int\limits_{\omega_k^2} K(x)dx\Big|\right\}.$$

We can assume that

$$\Big|\int\limits_{\omega_k^1} K(x)dx\Big| \geqslant \Big|\int\limits_{\omega_k^2} K(x)dx\Big|.$$

Let us consider two cases: $|\omega_k^1| \geqslant \frac{w}{2}$ and $|\omega_k^1| < \frac{w}{2}$. In the first case, there exists $\tilde{\omega}_k \subset \omega_k^1$ such that $|\tilde{\omega}_k| = \frac{w}{2}$ and

$$2\Big|\int\limits_{\tilde{\omega}_k} K(x)dx\Big| \geqslant \Big|\int\limits_{\omega_k^1} K(x)dx\Big|.$$

In the second case, $|\omega_k^1| < \frac{w}{2}$ and there exist η_k^1 and η_k^2 such that $|\eta_k^1 \bigcap \eta_k^2| = 0$, $\eta_k^1 \bigcup \eta_k^2 = \omega_k^2$, and $|\eta_k^i| = \frac{|\omega_k^2|}{2}$. Since $K(x)$ keeps its sign on ω_k^2, we have

$$\begin{aligned}\Big|\int\limits_{\omega_k^1} K(x)dx\Big| &\geqslant \Big|\int\limits_{\omega_k^2} K(x)dx\Big| = \Big|\int\limits_{\eta_k^1} K(x)dx\Big| + \Big|\int\limits_{\eta_k^2} K(x)dx\Big| \\ &\geqslant 2\min\left(\Big|\int\limits_{\eta_k^1} K(x)dx\Big|, \Big|\int\limits_{\omega_2^2} K(x)dx\Big|\right) = 2\Big|\int\limits_{\eta_k^{i_0}} K(x)dx\Big|.\end{aligned}$$

Here $\eta_k^{i_0}$ are sets where the infimum is attained. Then we consider $\eta_k \subset \eta_k^{i_0}$ such that $|\eta_k| = \frac{w}{2} - |\omega_k^1|$.

Let now $\tilde{\omega}_k = \eta_k \cup \omega_k^1$, then $|\tilde{\omega}_k| = \frac{w}{2}$ and

$$\begin{aligned}\Big|\int\limits_{\tilde{\omega}_k} K(x)dx\Big| &= \Big|\int\limits_{\omega_k^1} K(x)dx + \int\limits_{\eta_k} K(x)dx\Big| \\ &\geqslant \Big|\int\limits_{\omega_k^1} K(x)dx\Big| - \Big|\int\limits_{\eta_k} K(x)dx\Big| \\ &\geqslant \Big|\int\limits_{\omega_k^1} K(x)dx\Big| - \Big|\int\limits_{\eta_k^{i_0}} K(x)dx\Big| \geqslant \frac{1}{2}\Big|\int\limits_{\omega_1} K(x)dx\Big|.\end{aligned}$$

Therefore, in both cases, we have

$$\int\limits_{\omega_k} |K(x)|dx \leqslant 2\Big|\int\limits_{\omega_1} K(x)dx\Big| \leqslant 4\Big|\int\limits_{\tilde{\omega}_k} K(x)dx\Big|,$$

Suppose

$$J_+ = \{k : \int_{\tilde{\omega}_k} K(x)dx \geqslant 0\}, \qquad J_- = \{k : \int_{\tilde{\omega}_k} K(x)dx \leqslant 0\};$$

then

$$2\max\left\{\Big|\int_{\bigcup_{k\in J_+}\tilde{\omega}_k} K(x)dx\Big|, \Big|\int_{\bigcup_{k\in J_-}\tilde{\omega}_k} K(x)dx\Big|\right\}$$
$$\geqslant \sum_{k=1}^{m}\Big|\int_{\omega_k} K(x)dx\Big| \geqslant \frac{1}{4}\int_e |K(x)|dx.$$

Taking as e' the set $\bigcup_{k\in J_+}\tilde{\omega}_k$ or $\bigcup_{k\in J_-}\tilde{\omega}_k$, where the maximum is attained, we get

$$\frac{1}{|e|^\gamma}\int_e |K(x)|dx \leqslant 2^3\frac{1}{|e|^\gamma}\left|\int_{e'} K(x)dx\right| \leqslant 2^{3-\gamma}\frac{1}{|e'|^\gamma}\left|\int_{e'} K(x)dx\right|.$$

By construction, $\tilde{\omega}_k \subset \omega_k$ and $|\tilde{\omega}_k| = \frac{\omega}{2}$, and therefore $e' \in \mathfrak{U}(I)$. □

4. Proof of lower bound for $\|A\|_{L_p\to L_q}$

Let $1 < p < q < \infty$, $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$, and $Af = K * f$ is bounded from $L_p(\mathbb{R})$ in $L_q(\mathbb{R})$. We are going to prove that for any number d and an interval I, $|I| = d$, there holds

$$\sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right| \leqslant c(p,q)\|A\|_{L_p\to L_q}, \tag{4.1}$$

where the collection $\mathfrak{L}(I)$ is given by (1.4). We define $\mathfrak{L}'(I) \subset \mathfrak{L}(I)$ as follows

$$\mathfrak{L}'(I) = \left\{e = \bigcup_{k=0}^{m}([a,b] + kd) : m \in \mathbb{N}, [a,b] \subset I, b - a \leqslant d/2\right\}.$$

Note that for any locally summable function $K(x)$ we have

$$\sup_{e\in\mathfrak{L}'(I)} \frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right| \leqslant \sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|$$
$$\leqslant 2^{1/r}\sup_{e\in\mathfrak{L}'(I)} \frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|.$$

Indeed, the left-hand side inequality is clear since $\mathfrak{L}'(I)\subset\mathfrak{L}(I)$. To prove the right-hand side, we consider $e\in\mathfrak{L}(I)$, that is, $e=\bigcup_{i=0}^m([a,b]+id)=\bigcup_{i=0}^m([a,\frac{a+b}{2}]+id)$ $\cup\bigcup_{i=0}^m([\frac{a+b}{2},b]+id)=e_1\cup e_2$. Clearly, $|e_i|=|e|/2$, $e_i\in\mathfrak{L}'(I)$, $i=1,2$. Therefore,

$$\begin{aligned}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right| &\leqslant \frac{2}{|e|^{1/r'}}\max_{i=1,2}\left|\int_{e_i}K(x)dx\right| \\ &= 2^{1-1/r'}\max_{i=1,2}\frac{1}{|e_i|^{1/r'}}\left|\int_{e_i}K(x)dx\right| \\ &\leqslant 2^{1/r}\sup_{e\in\mathfrak{L}'(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|.\end{aligned}$$

Hence, it is sufficient to verify

$$\sup_{e\in\mathfrak{L}'(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|\leqslant c\|A\|_{L_p\to L_q}.$$

Let us first assume that K is bounded, that is, $|K(x)|\leqslant D$, $x\in\mathbb{R}$. For $s>0$ we define

$$\alpha_s=\sup_{\substack{e\in\mathfrak{L}'(I)\\ |e|\leqslant s}}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|.$$

This is well-defined since for any $e\in\mathfrak{L}'(I)$ and $|e|\leqslant s$ we get

$$\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|\leqslant D\,|e|^{1-1/r'}\leqslant Ds^{1-1/r'}.$$

Then we consider $e_0\in\mathfrak{L}'(I)$, $|e_0|\leqslant s$ such that

$$\frac{1}{|e_0|^{1/r'}}\left|\int_{e_0}K(x)dx\right|\geqslant\frac{\alpha_s}{2}.$$

Since the convolution is translation invariant, then we assume that e_0 is of form

$$e_0=\bigcup_{i=0}^m([0,b]+id)\,,$$

where $b\leqslant d/2$, $m\in\mathbb{N}\cup\{0\}$.

Let us take $0<\delta<\frac{1}{2}$ to be specified later. We define the following sets $e_{1+\delta}$ and e_δ:

$$e_{1+\delta}=\bigcup_{i=0}^{[(1+\delta)m]}([0,(1+\delta)b]+id)\,,$$

$$e_\delta=\bigcup_{i=0}^{[\delta m]}([0,\delta b]+id)\,.$$

Since $e_0 \in \mathfrak{L}'(I)$, we have $e_{1+\delta} \in \mathfrak{L}(I)$. Then taking $f_0 = \chi_{e_{1+\delta}}$, boundedness of the operator A implies

$$\begin{aligned} \|K * f_0\|_{L_q} &\leqslant \|A\|_{L_p \to L_q} \|f_0\|_{L_p} \\ &= \|A\|_{L_p \to L_q} |e_{1+\delta}|^{1/p} \leqslant 2 \|A\|_{L_p \to L_q} (1+\delta)^{2/p} |e_0|^{1/p}. \end{aligned} \tag{4.2}$$

On the other hand,

$$\begin{aligned} \|K * f_0\|_{L_q} &= \left(\int_{-\infty}^{\infty} \left| \int_{e_{1+\delta}} K(x-y)dx \right|^q dy \right)^{1/q} \\ &= \left(\sum_{j \in \mathbb{Z}} \int_0^d \left| \sum_{i=0}^{(1+\delta)m} \int_0^{b(1+\delta)} K((i-j)d + (x-y))dx \right|^q dy \right)^{1/q} \\ &\geqslant \left(\sum_{j=0}^{[\delta m]} \int_0^{\delta b} \left| \sum_{i=0}^{(1+\delta)m} \int_0^{(1+\delta)b} K((i-j)d + (x-y))dx \right|^q dy \right)^{1/q} \\ &= \left(\sum_{j=0}^{[\delta m]} \int_0^{\delta b} \left| \sum_{i=-j}^{(1+\delta)m-j} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right|^q dy \right)^{1/q} \\ &\geqslant \left(\sum_{j=0}^{[\delta m]} \int_0^{\delta b} \left[\left| \sum_{i=0}^{m} \int_0^{b} K(id+x)dx \right| - \left| \sum_{i=-j}^{-1} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right| \right. \right. \\ &\quad - \left| \sum_{i=m+1}^{[(1+\delta)m]-j} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right| - \left| \sum_{i=0}^{m} \int_{-y}^{0} K(id+x)dx \right| \\ &\quad \left. \left. - \left| \sum_{i=0}^{m} \int_{b}^{(1+\delta)b-y} K(id+x)dx \right| \right]^q dy \right)^{1/q} \\ &=: \left(\int_{e_\delta} \left[\left| \int_{e_0} K(x)dx \right| - \sum_{i=1}^{4} \left| \int_{e_i} K(x)dx \right| \right]^q dy \right)^{1/q}, \end{aligned}$$

where $e_i \in \mathfrak{L}'(I)$ such that $|e_i| \leqslant 2\delta |e_0|$, $i = 1, 2, 3, 4$.

We put $\delta = \left(2(16^{r'})\right)^{-1} < \frac{1}{2}$. Then $|e_i| < |e_0| \leqslant s$ and

$$\frac{1}{|e_0|^{1/r'}} \left| \int_{e_0} K(x)dx \right| \geqslant \frac{\alpha_s}{2} \geqslant \frac{1}{2\,|e_i|^{1/r'}} \left| \int_{|e_i|} K(x)dx \right|$$

and therefore

$$\left|\int_{e_i} K(x)dx\right| \leqslant \frac{2\,|e_i|^{1/r'}}{|e_0|^{1/r'}} \left|\int_{e_0} K(x)dx\right|.$$

Taking into account $|e_i| \leqslant 2\delta\,|e_0|$, we get

$$\begin{aligned} \|K * f_0\|_{L_q} &\geqslant \left(\int_{e_\delta} \left[\left|\int_{e_0} K(x)dx\right| \left(1 - 2\sum_{i=1}^{4}\left(\frac{|e_i|}{|e_0|}\right)^{1/r'}\right)\right]^q dy\right)^{1/q} \\ &\geqslant |e_\delta|^{1/q} \left|\int_{e_0} K(x)dx\right| \left(1 - 8(2\delta)^{1/r'}\right) \\ &\geqslant \frac{1}{2}\delta^{2/q}\,|e_0|^{1/q} \left|\int_{e_0} K(x)dx\right|. \end{aligned}$$

Using (4.2), we have

$$\|A\|_{L_p\to L_q} \geqslant C_{p,q}\frac{1}{|e_0|^{1/r'}} \left|\int_{e_0} K(x)dx\right| \geqslant \frac{C_{p,q}}{2} \sup_{\substack{e\in\mathfrak{L}'(I)\\ |e|\leqslant s}} \frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right|$$

Thus, for the bounded K and for any $s > 0$ we obtain

$$\sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| \leqslant C\,\|A\|_{L_p\to L_q}, \tag{4.3}$$

where C depends on p and q.

To prove this in a general case of locally integrable K not necessary bounded, we consider

$$K_N(x) = \begin{cases} N, & K(x) > N, \\ K(x), & K(x) \leqslant N \end{cases}, \qquad N \in \mathbb{N}$$

and

$$K_{N,M}(x) = \begin{cases} K_N(x) & K(x) \geqslant -M, \\ M & K(x) < -M \end{cases}, \qquad N, M \in \mathbb{N}.$$

As we have proved before,

$$\sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/r'}} \left|\int_e K_{N,M}(x)dx\right| \leqslant C\,\|A_{N,M}\|_{L_p\to L_q}, \qquad A_{N,M}f = K_{N,M} * f,$$

where a constant C does not depend on N and M.

Noting that Banach-Steinhaus' theorem implies $\|A_{N,M}\|_{L_p\to L_q} \leqslant D(D > 0)$ for some $D > 0$ and using the monotonicity properties of $K_{N,M}$, namely,

$$K_{N,1}(x) \geqslant K_{N,2} \geqslant \ldots \geqslant K_{N,M} \geqslant \ldots$$

and

$$K_1(x) \leqslant K_2(x) \leqslant \ldots K_N \leqslant \ldots,$$

we apply Levi's theorem:

$$\sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| \leqslant cD < \infty.$$

Finally, repeating the proof of (4.3), we arrive at required inequality (4.1). □

We would like to mention that attempts have already been made at proving the lower estimate for the convolution operator in [NS], although they require stronger hypotheses than those used here.

5. Comparison with O'Neil and Stepanov's inequalities

Let us first show that the right-hand side estimate in (1.5) implies both (1.2) and (1.3). Indeed, it is known that ([BS, Ch. 2, §3])

$$\sup_{t>0} t^{1/r}K^*(t) \approx \sup_{t>0} t^{1/r}K^{**}(t) \approx \sup_{0<|e|<\infty} \frac{1}{|e|^{1/r'}} \int_e |K(x)|\, dx, \tag{5.1}$$

and therefore

$$\sup_{e\in\mathfrak{U}(I)} \frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| \leqslant C \sup_{t>0} t^{1/r}K^*(t).$$

Let $1/r = 1-(1/p-1/q) < 1$, $r' = r/(r-1)$, and let I be an interval with $|I| = 1$. Assume that $e \in \mathfrak{U}(I)$, that is, $e = \bigcup_{s=1}^m w_s$ such that $w_s \in I + i_s, |w_s| = |w_j| = g \leqslant 1, i_s \neq i_j$. Then

$$\begin{aligned}
\frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| &= \left|\frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \int_{w_s} K(x)dx\right| \\
&= \frac{1}{(mg)^{1/r'}} \left|\sum_{s=1}^m \int_0^1 K(x+i_s)\chi_{w_s}(x+i_s)dx\right| \\
&\leqslant \frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \int_0^1 |\tilde{K}(x,i_s)\tilde{\chi}_{w_s}(x,i_s)|dx \\
&= \frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \int_0^g \tilde{K}^*(t,i_s)dt \\
&\leqslant \frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \sup_{0<t\leqslant 1} t^{1/r}\tilde{K}^*(t,i_s) \int_0^g t^{-1/r}dt \\
&\leqslant (r')^2 \sup_{n\in\mathbb{N}} s^{1/r} \left(\sup_{0<t\leqslant 1} t^{1/r}\tilde{K}^*(t,\cdot)\right)_s^* .
\end{aligned}$$

Thus, (1.5) is stronger than either (1.2) or (1.3). We now give an example capturing the difference between these estimates.

Example. *Let $1 < p < q < \infty$ and $0 < 1/r = 1 - (1/p - 1/q)$. Define the function $K(x)$ on $\mathbb{R}$ as follows:*

$$K(x) = \begin{cases} 2^{k/r}, & \text{for } x \in [-k, -k + 2^{-k}], \quad k \in \mathbb{N}; \\ 1, & \text{for } x \in [k, k + 1/k), \quad k \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

This function satisfies

$$\sup_{e \in \mathfrak{U}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K(x) dx \right| < \infty, \tag{5.2}$$

but

$$\|K\|_{L_{r,\infty}} = \infty \tag{5.3}$$

and

$$\|K\|_{W(L_{r,\infty}[0,1],\, l_{r,\infty}(\mathbb{Z}))} = \sup_{n \in \mathbb{N}} n^{1/r} \left(\sup_{0 \leqslant t \leqslant 1} t^{1/r} \tilde{K}^*(t, \cdot) \right)^*_n = \infty. \tag{5.4}$$

Indeed, let us show (5.2). Let $K_+(x) = K(x)\chi_{[0,\infty)}(x)$, $K_-(x) = K(x)\chi_{(-\infty,0)}(x)$, then $K_+(x) + K_-(x) = K(x)$ and therefore,

$$\sup_{e \in M([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K(x) dx \right| \leqslant \sup_{e \in M([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K_+(x) dx \right| + \sup_{e \in M([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K_-(x) dx \right|.$$

Let $e \in \mathfrak{U}([0,1])$. Then $e = \cup_{k \in u} w_k$, where $|w_k| = w < 1$ and $u \subset \mathbb{Z}$, $|u| = m$. We have

$$\begin{aligned} \frac{1}{|e|^{1/r'}} \int_e K_+(x)\, dx &= \frac{1}{|m|^{1/r'}} \frac{1}{w^{1/r'}} \sum_{k \in u} \left| \int_{w_k} K_+(x) dx \right| \\ &= \frac{1}{m^{1/r'}} \frac{1}{w^{1/r}} \sum_{n=1}^{m} \int_0^w K_+(x+n) dx \\ &= \frac{1}{(wm)^{1/r'}} \left(\sum_{n=1}^{1/w} \int_0^w K_+(x+n) dx + \sum_{n=1/w}^{m} \int_0^{1/n} K_+(x+n) dx \right) \\ &= \frac{1}{(wm)^{1/r'}} \left(\sum_{n=1}^{1/w} w + \sum_{n=1/w}^{n} \frac{1}{n} \right) \\ &\leqslant \frac{2}{(wm)^{1/r'}} \big(1 + \ln(mw)\big) \leqslant 2r'. \end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{|e|^{1/r'}}\int_e K_-(x)\,dx &= \frac{1}{|m|^{1/r'}}\frac{1}{w^{1/r'}}\sum_{k\in u}\left|\int_{w_k}K_-(x)dx\right|\\ &= \frac{1}{(mw)^{1/r'}}\frac{1}{w^{1/r}}\sum_{n\in u}\int_0^w K_-(x+n)dx\\ &\leqslant \frac{1}{(wm)^{1/r'}}\left(\sum_{\substack{n\in u\\ |n|<\log_2\frac{1}{w}}}\int_0^w K_-(x-|n|)dx+\right.\\ &\qquad\left.\sum_{\substack{n\in u\\ |n|\geqslant\log_2\frac{1}{w}}}\int_0^{2^{-|n|}} K_-(x-|n|)dx\right)\\ &= \frac{1}{(wm)^{1/r'}}\left(\sum_{\substack{n\in u\\ |n|<\log_2\frac{1}{w}}}2^{|n|/r}w+\sum_{\substack{n\in u\\ |n|\geqslant\log_2\frac{1}{w}}}2^{-|n|/r'}\right)\\ &\leqslant 2\frac{1}{(wm)^{1/r'}}\left(w^{1/r'}+w^{1/r'}\right)\leqslant 4.\end{aligned}$$

Combining these estimates, we get

$$\sup_{e\in\mathfrak{U}([0,1])}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|\leqslant 4+2r'.$$

To show (5.3), we note that $K_+^*(t)\equiv 1$. Hence,

$$\sup_{t>0}\ t^{1/r}K^*(t)\geqslant\sup_{t>0}\ t^{1/r}K_+^*(t)=\infty.$$

To show (5.4), we note

$$\begin{aligned}\|K\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))} &\geqslant \|K_-\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))}\\ &= \sup_{n\in\mathbb{N}} n^{1/r}\left(\sup_{0<t\leqslant 1} t^{1/r}(\widetilde{K_N^-})^*(t,n)\right)_n^*\\ &= \sup_{n\in\mathbb{N}} n^{1/r}\left(\sup_{0<t\leqslant 2^{-n}} t^{1/r}2^{n/r}\right)\\ &= \sup_{n\in\mathbb{N}} n^{1/r}=\infty.\end{aligned}$$

□

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E. Nursultanov
Kazakh Branch of Moscow State University
Munatpasova, 7
010010 Astana Kazakhstan
E-mail address: er-nurs@yandex.ru

S. Tikhonov
ICREA and Centre de Recerca Matemàtica
Apartat 50 08193 Bellaterra, Barcelona, Spain
E-mail address: stikhonov@crm.cat

N. Tleukhanova
Gumilyov Eurasian National University
Munatpasova, 5
010008 Astana Kazakhstan
E-mail address: tleukhanova@rambler.ru