Mathematical Notes, vol. 75, no. 3, 2004, pp. 341–351.
Translated from Matematicheskie Zametki, vol. 75, no. 3, 2004, pp. 372–383.

Application of Interpolational Methods to the Study of Properties of Functions of Several Variables

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Received November 18, 1998; in final form, April 2, 2003

Abstract—In this paper, interpolation theorems for spaces of functions of several variables are used to generalize and refine Hörmander's theorem on the multipliers of the Fourier transform from L_p to L_q and the Hardy–Littlewood–Paley inequality for a class of multiple Fourier series in the multidimensional case.

Key words: *function of several variables, interpolation theorem, Fourier transform, Hardy–Littlewood–Paley inequality, Banach space.*

The idea of "multidimensional" interpolation of Banach spaces originated in the papers of Yoshikawa [1], and Sparr [2] and was further developed in [3–10].

In this paper, we intend to show how interpolation theorems for spaces of functions of several variables [9, 10] can be used to generalize and refine well-known results.

1. Suppose that $\mathbf{p}=(p_1,\dots,p_n)$ and $\mathbf{q}=(q_1,\dots,q_n)$ are vectors such that if $0<q_j<\infty$, then $0<p_j<\infty$, but if $q_j=\infty$, then $0<p_j\le\infty$, $j=1,\dots,n$. In what follows, we assume that the vectors $\mathbf{p}$ and $\mathbf{q}$ satisfy this conditions.

We define the functionals

$$\Phi_{\mathbf{pq}}(\varphi)=\left(\int_0^\infty\dots\left(\int_0^\infty\left|t_1^{\frac{1}{p_1}}\dots t_n^{\frac{1}{p_n}}\varphi(t_1,\dots,t_n)\right|^{q_1}\frac{dt_1}{t_1}\right)^{q_2/q_1}\dots\frac{dt_n}{t_n}\right)^{1/q_n},$$

$$F_{\mathbf{pq}}(a)=\left(\sum_{k_n=1}^\infty k_n^{\frac{q_n}{p_n}-1}\dots\left(\sum_{k_1=1}^\infty k_1^{\frac{q_1}{p_1}-1}|a_{k_1\dots k_n}|^{q_1}\right)^{q_2/q_1}\dots\right)^{1/q_n},$$

where the expressions

$$\left(\int_0^\infty(G(t))^q\frac{dt}{t}\right)^{1/q}\qquad\text{and}\qquad\left(\sum_{k=1}^\infty|g_k|^q\right)^{1/q}\qquad\text{for}\quad q=\infty$$

are understood as $\sup_{t>0}G(t)$ and $\sup_k|g_k|$, respectively.

Suppose that $\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n$, $f(x_1,\dots,x_n)$ is a measurable function given in $\mathbb{R}^{\mathbf{m}}=\mathbb{R}^{m_1}\times\dots\times\mathbb{R}^{m_n}$, and $*=(j_1,\dots,j_n)$ is some rearrangement of the sequence of numbers $\{1,2,\dots,n\}$. By $f^*(t)=f^{*_{j_1},\dots,*_{j_n}}(t_1,\dots,t_n)$ we denote the function resulting from the application of a nonincreasing rearrangement successively to the variables $x_{j_1}\in\mathbb{R}^{m_{j_1}},\dots,x_{j_n}\in\mathbb{R}^{m_{j_n}}$, assuming the other variables fixed. This function $f^*(t)$ is called the *nonincreasing rearrangement* of the function f in $\mathbb{R}^{\mathbf{m}}$ corresponding to the vector $*=(j_1,\dots,j_n)$.

The space $L^*_{\mathbf{pq}}(\mathbb{R}^{\mathbf{m}})$ is defined as the set of functions for which

$$\|f\|_{L^*_{\mathbf{pq}}(\mathbb{R}^{\mathbf{m}})} = \Phi_{\mathbf{pq}}(f^*(\cdot)) < \infty.$$

If $* = (1, 2, \dots, n)$, then the space $L^*_{\mathbf{pq}}$ is denoted as $L_{\mathbf{pq}}$.

The space $L^*_{\mathbf{pq}}(\Omega)$, where $\Omega = \Omega_1 \times \dots \times \Omega_n$, $\Omega_k \subset \mathbb{R}^{m_k}$, is defined by the norm

$$\|f\|_{L^*_{\mathbf{pq}}(\Omega)} = \|f_\Omega\|_{L^*_{\mathbf{pq}}(\mathbb{R}^{\mathbf{m}})},$$

where f_Ω is the zero continuation of the function f from the domain Ω to the whole space $\mathbb{R}^{\mathbf{m}}$.

Similarly, we define the space

$$l^*_{\mathbf{pq}} = \bigl\{\{a_s\}_{s\in\mathbb{N}^n} \colon \|a\|_{l^*_{\mathbf{pq}}} = F_{\mathbf{pq}}(\{a^*_k\}) < \infty\bigr\},$$

where $a^*_{\mathbf{k}} = a^{*_{j_1}\cdots *_{j_n}}_{k_1,\dots,k_n}$ is the nonincreasing rearrangement of the sequence $\{a_s\}_{s\in\mathbb{N}^n}$ that was applied successively to the variables $s_{j_1}, \dots, s_{j_n}$.

The spaces $L^*_{\mathbf{pq}}$, $l^*_{\mathbf{pq}}$ are a generalization of the Lorentz spaces to the anisotropic case. For the case in which $* = (1, 2, \dots, n)$, $\mathbf{m} = (1, \dots, 1)$, these spaces were studied by Blozinski [7] and later some of their properties were investigated. In the general case, these properties are preserved. (see [9]).

(a) For $\mathbf{q} \le \mathbf{q_1}$ $(q_j \le q^1_j,\ \ j = 1, \dots, n)$, the following embeddings are valid:

$$L^*_{\mathbf{pq}}(\Omega) \hookrightarrow L^*_{\mathbf{pq_1}}(\Omega), \qquad l^*_{\mathbf{pq}} \hookrightarrow l^*_{\mathbf{pq_1}}.$$

(b) If $p_{j_0} < \infty$ and the domain $\Omega = \Omega_1 \times \dots \times \Omega_n$ satisfies $m\Omega_{j_0} < \infty$, then for any $\varepsilon > 0$ the following embeddings hold:

$$L^*_{\mathbf{p}_\varepsilon\mathbf{q}_1}(\Omega) \hookrightarrow L^*_{\mathbf{pq}}(\Omega), \qquad l^*_{\mathbf{pq_1}} \hookrightarrow l^*_{\mathbf{p}_\varepsilon\mathbf{q}},$$

where $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n)$, $\mathbf{p}_\varepsilon = (p_1, \dots, p_{j_0-1}, p_{j_0} + \varepsilon, p_{j_0+1}, \dots, p_n)$, and $\mathbf{q_1} = (q_1, \dots, q_{j_0-1}, \infty, q_{j_0+1}, \dots, q_n)$.

(c) (Hölder's inequality). Suppose that $\mathbf{1} < \mathbf{p}, \mathbf{r}, \mathbf{q} < \infty$, $\mathbf{1} \le \mathbf{t}, \mathbf{s}, \boldsymbol{\nu} \le \infty$, $\mathbf{1/p} + \mathbf{1/r} = \mathbf{1/q}$, $\mathbf{1/s} + \mathbf{1/t} \ge \mathbf{1}/\boldsymbol{\nu}$ $(1/p_j + 1/r_j = 1/q_j,\ \ 1/s_j + 1/t_j \ge 1/\nu_j,\ \ j = 1, \dots, n)$; then the following inequalities are valid:

$$\|fg\|_{L^*_{\mathbf{q}\boldsymbol{\nu}}(\Omega)} \le c\|f\|_{L^*_{\mathbf{ps}}(\Omega)}\|g\|_{L^*_{\mathbf{rt}}(\Omega)}, \qquad \|ab\|_{l^*_{\mathbf{q}\boldsymbol{\nu}}} \le c\|a\|_{l^*_{\mathbf{ps}}}\|b\|_{l^*_{\mathbf{rt}}}.$$

Suppose that A_1 is a Banach space and A_2 is a functional Banach lattice. By $A = (A_1, A_2)$ we denote the space of A_1-valued measurable functions such that $\|f(x)\|_{A_1} \in A_2$ with norm $\|f\| = \|\|f(x)\|_{A_1}\|_{A_2}$. The space $\mathbf{A} = (A_1, \dots, A_n)$ is defined inductively. We call it an *anisotropic space of dimension* n.

Suppose that $\mathbf{A}_0 = (A^0_1, \dots, A^0_n)$, $\mathbf{A}_1 = (A^1_1, \dots, A^1_n)$ are two anisotropic spaces, while $E = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \colon \varepsilon_j = 0 \text{ or } \varepsilon_j = 1,\ \ j = 1, \dots, n\}$ are the vertices of the n-dimensional unit cube. For an arbitrary $\varepsilon \in E$, we define the space $\mathbf{A}_\varepsilon = (A^{\varepsilon_1}_1, \dots, A^{\varepsilon_n}_n)$ with norm

$$\|a\|_{\mathbf{A}_\varepsilon} = \|\dots\|a\|_{A^{\varepsilon_1}_1}\dots\|_{A^{\varepsilon_n}_n}.$$

A pair of anisotropic spaces $\mathbf{A}_0 = (A^0_1, \dots, A^0_n)$, $\mathbf{A}_1 = (A^1_1, \dots, A^1_n)$ is said to be *consistent* if there exists a linear Hausdorff space containing $\mathbf{A}_\varepsilon$, $\varepsilon \in E$ as its subsets.

Suppose that $* = (j_1, \dots, j_n)$ is some rearrangement of the sequence $(1, 2, \dots, n)$. To the vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in E$ we assign $\varepsilon^* = (\varepsilon_{j_1}, \dots, \varepsilon_{j_n}) \in E$. and define the functional K^*:

$$K^*(t, a\,; \mathbf{A}_0, \mathbf{A}_1) = \inf\biggl\{\sum_{\varepsilon\in E} t^\varepsilon\|a_{\varepsilon^*}\|_{\mathbf{A}_{\varepsilon^*}} : a = \sum_{\varepsilon\in E} a_\varepsilon, \quad a_\varepsilon \in \mathbf{A}_\varepsilon\biggr\},$$

where $t^\varepsilon = t^{\varepsilon_1}_1 \dots t^{\varepsilon_n}_n$.

Suppose that $\mathbf{0} < \boldsymbol{\theta} = (\theta_1, \dots, \theta_n) < \mathbf{1}$ and $\mathbf{0} < \mathbf{q} = (q_1, \dots, q_n) \le \infty$. By $\mathbf{A}^*_{\boldsymbol{\theta}\mathbf{q}} = (\mathbf{A}_0, \mathbf{A}_1)^*_{\boldsymbol{\theta}\mathbf{q}}$ we denote a linear subset $\sum_{\varepsilon\in E}\mathbf{A}_\varepsilon$ whose elements satisfy

$$\|a\|_{\mathbf{A}^*_{\boldsymbol{\theta}\mathbf{q}}} = \Phi_{\mathbf{pq}}(t^{-1}K^*(t, a)) < \infty,$$

where $\mathbf{p} = \mathbf{1}/(\mathbf{1} - \boldsymbol{\theta})$.

Lemma 1. *Let* $(\mathbf{A}_0, \mathbf{A}_1)$ *be a consistent pair of anisotropic spaces,*

$$E = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) : \ \varepsilon_j = 0 \text{ or } \varepsilon_j = 1, \ j = 1, \dots, n\}.$$

Suppose that $* = (j_1, \dots, j_n)$ *and* $*_1 = (r_1, \dots, r_n)$ *are some rearrangements of the sequence* $(1, \dots, n)$ *and* $* \circ *_1 = (r_{j_1}, \dots, r_{j_n})$ *defines the rearrangement of the sequence* $(r_1, \dots, r_n)$ *corresponding to the vector* $* = (j_1, \dots, j_n)$. *If* T *is a linear operator such that* $T \colon \mathbf{A}_{\varepsilon^*} \to \mathbf{B}_\varepsilon$ *with norm* M_ε *for any* $\varepsilon \in E$, *then*

$$T \colon \mathbf{A}_{\boldsymbol{\theta}\mathbf{q}}^{*\circ *_1} \to \mathbf{B}_{\boldsymbol{\theta}\mathbf{q}}^{*_1}, \qquad \mathbf{0} < \boldsymbol{\theta} < \mathbf{1}, \qquad \mathbf{0} < \mathbf{q} \le \infty,$$

with norm $\|T\| \le \max_{\varepsilon \in E} M_\varepsilon$.

The following assertion is a consequence of Theorems 1 and 2 from [9].

Theorem 1. *Suppose that* $\mathbf{1} \le \mathbf{p_i} = (p_1^i, \dots, p_n^i)$, $\boldsymbol{\sigma_i} = (\sigma_1^i, \dots, \sigma_n^i) \le \boldsymbol{\infty}$, $i = 0, 1$, $\mathbf{p_0} \ne \mathbf{p_1}$, $\mathbf{0} < \mathbf{q} = (q_1, \dots, q_n) \le \infty$, $\mathbf{0} < \boldsymbol{\theta} = (\theta_0, \dots, \theta_n) < \mathbf{1}$, $\mathbf{1/p} = (\mathbf{1} - \boldsymbol{\theta})/\mathbf{p_0} + \boldsymbol{\theta}/\mathbf{p_1}$, $* = (j_1, \dots, j_n)$ *is a rearrangement of the sequence* $\{1, 2, \dots, n\}$, *and*

$$\mathbf{A}^*_{\mathbf{p_0}\boldsymbol{\sigma_0}} = \bigl(L_{p^0_{j_1}\sigma^0_{j_1}}, \dots, L_{p^0_{j_n}\sigma^0_{j_n}}\bigr), \qquad \mathbf{A}^*_{\mathbf{p_1}\boldsymbol{\sigma_1}} = \bigl(L_{p^1_{j_1}\sigma^1_{j_1}}, \dots, L_{p^1_{j_n}\sigma^1_{j_n}}\bigr),$$
$$\mathbf{B}^*_{\mathbf{p_0}\boldsymbol{\sigma_0}} = \bigl(l_{p^0_{j_1}\sigma^0_{j_1}}, \dots, l_{p^0_{j_n}\sigma^0_{j_n}}\bigr), \qquad \mathbf{B}^*_{\mathbf{p_1}\boldsymbol{\sigma_1}} = \bigl(l_{p^1_{j_1}\sigma^1_{j_1}}, \dots, l_{p^1_{j_n}\sigma^1_{j_n}}\bigr).$$

Then[1]

$$L^*_{\mathbf{pq}} = (\mathbf{A}^*_{\mathbf{p_0}\boldsymbol{\sigma_0}}, \mathbf{A}^*_{\mathbf{p_1}\boldsymbol{\sigma_1}})^*_{\theta\mathbf{q}}, \qquad l^*_{\mathbf{pq}} = (\mathbf{B}^*_{\mathbf{p_0}\boldsymbol{\sigma_0}}, \mathbf{B}^*_{\mathbf{p_1}\boldsymbol{\sigma_1}})^*_{\theta\mathbf{q}},$$
$$L^*_{\mathbf{pq}} = (L^*_{\mathbf{p_0}\boldsymbol{\infty}}, L^*_{\mathbf{p_1}\boldsymbol{\infty}})^*_{\theta\mathbf{q}}, \qquad l^*_{\mathbf{pq}} = (l^*_{\mathbf{p_0}\boldsymbol{\infty}}, l^*_{\mathbf{p_1}\boldsymbol{\infty}})^*_{\theta\mathbf{q}}.$$

Remark. By Theorem 1, the order (integration) of the spaces in $\mathbf{A}^*_{\mathbf{p_i}\boldsymbol{\sigma_i}} = \bigl(L_{p^i_{j_1}\sigma^i_{j_1}}, \dots, L_{p^i_{j_n}\sigma^i_{j_n}}\bigr)$, $i = 0, 1$, determines the order of nonincreasing rearrangement in $L^*_{\mathbf{pq}} = (\mathbf{A}^*_{\mathbf{p_0}\boldsymbol{\sigma_0}}, \mathbf{A}^*_{\mathbf{p_1}\boldsymbol{\sigma_1}})^*_{\theta\mathbf{q}}$. Thus, interpolation by the same method of the pair of spaces $\bigl(L_{p_0}(\mathbb{R}_x), L_{p_0}(\mathbb{R}_y)\bigr)$, $\bigl(L_{p_1}(\mathbb{R}_x), L_{p_1}(\mathbb{R}_y)\bigr)$ and $\bigl(L_{p_0}(\mathbb{R}_y), L_{p_0}(\mathbb{R}_x)\bigr)$, $\bigl(L_{p_1}(\mathbb{R}_y), L_{p_1}(\mathbb{R}_x)\bigr)$ yields two different results, although these pairs coincide as linear normed spaces:

$$L_{p_i}(\mathbb{R}^2) = \bigl(L_{p_i}(\mathbb{R}_x), L_{p_i}(\mathbb{R}_y)\bigr) = \bigl(L_{p_i}(\mathbb{R}_y), L_{p_i}(\mathbb{R}_x)\bigr), \qquad i = 0, 1.$$

The order integration in $L^*_{\mathbf{pq}} = (\mathbf{A}^*_{\mathbf{p_0}\boldsymbol{\sigma_0}}, \mathbf{A}_{\mathbf{p_1}\boldsymbol{\sigma_1}})^*_{\theta\mathbf{q}}$ is defined by the interpolation method.

2. Suppose that F and F^{-1} are the Fourier transform and its inverse, respectively, in $\mathbb{R}^n$. A function φ is called a *multiplier of the Fourier transform* from the functional Lorentz space $L_{\mathbf{p}\boldsymbol{\tau}}(\mathbb{R}^n)$ to the Lorentz space $L_{\mathbf{qs}}(\mathbb{R}^n)$ if the operator $T_\varphi(f) = F^{-1}\varphi F f$ from the space $L_{\mathbf{p}\boldsymbol{\tau}}(\mathbb{R}^n)$ to $L_{\mathbf{qs}}(\mathbb{R}^n)$ is bounded. The set of all multipliers from $L_{\mathbf{p}\boldsymbol{\tau}}$ to $L_{\mathbf{qs}}$ is denoted by $M(L_{\mathbf{p}\boldsymbol{\tau}} \to L_{\mathbf{qs}})$. This set is a linear normed space with norm

$$\|\varphi\|_{M(L_{\mathbf{p}\boldsymbol{\tau}} \to L_{\mathbf{qs}})} = \sup_{f \ne 0} \frac{\|T_\varphi(f)\|_{L_{\mathbf{qs}}}}{\|f\|_{L_{\mathbf{p}\boldsymbol{\tau}}}}.$$

[1] Here $L_{p^i_{j_s}\sigma^i_{j_s}}$ and $l_{p^i_{j_s}\sigma^i_{j_s}}$ are ordinary Lorentz spaces (with scalar parameters).

Theorem 2. *Suppose that F is the Fourier transform,* $\mathbf{2} < \mathbf{p} = (p_1, \dots, p_n) < \infty$, $\mathbf{p}' = \mathbf{p}/(\mathbf{p}-\mathbf{1})$, $\mathbf{0} < \mathbf{q} = (q_1, \dots, q_n) \le \infty$, *and* $* = (j_1, \dots, j_n)$ *is a rearrangement of the sequence* $(1, 2, \dots, n)$, $*' = (j_n, \dots, j_1)$. *Then* $\|Ff\|_{L^*_{\mathbf{pq}}(\mathbb{R}^n)} \le c\|f\|_{L^{*'}_{\mathbf{p'q}}(\mathbb{R}^n)}$.

Proof. The proof is carried out for the case in which $* = (1, 2, \dots, n)$. Suppose that $\mathbf{p}_0 = (p_1^0, \dots, p_n^0) = (2, \dots, 2)$, $\mathbf{p}_1 = (p_1^1, \dots, p_n^1) = (\infty, \dots, \infty)$, and $E = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)\colon \varepsilon_i = 0$ or $\varepsilon_i = 1,\ i = 1, \dots, n\}$. We introduce the notation $\mathbf{p}_\varepsilon = (p_{\varepsilon_1}, \dots, p_{\varepsilon_n}) = (p_1^{\varepsilon_1}, \dots, p_n^{\varepsilon_n})$. Let us show that for any $\varepsilon \in E$ the following inequality holds:

$$\|Ff\|_{L_{\mathbf{p}_\varepsilon}} \le \|\dots\|\|f\|_{L_{p'_{\varepsilon_n}}}\|_{L_{p'_{\varepsilon_{n-1}}}} \dots\|_{L_{p'_{\varepsilon_1}}}; \tag{1}$$

here $1/p'_{\varepsilon_j} + 1/p_{\varepsilon_j} = 1$ $j = 1, \dots, n$. For $n = 1$, the proof follows from Parseval's relation and the definition of the Fourier transform. Suppose that inequality (1) is valid for n. Let us show that it is also valid for $n+1$. Let $\varepsilon_{n+1} = 0$ i.e., $p_{\varepsilon_{n+1}} = 2$, $\mathbf{p}_\varepsilon = (p_{\varepsilon_1}, \dots, p_{\varepsilon_n})$, $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$. Using the induction assumption, we obtain

$$\begin{aligned}
\|Ff\|^2_{L_{(\mathbf{p}_\varepsilon, p_{\varepsilon_{n+1}})}(\mathbb{R}^{n+1})} &= \int_{-\infty}^{\infty} \|Ff\|^2_{L_{\mathbf{p}_\varepsilon}(\mathbb{R}^n)}\, dy_{n+1} \\
&= \int_{-\infty}^{\infty} \bigg\| \int_{\mathbb{R}^n} \bigg(\int_{\infty}^{\infty} f(x, x_{n+1}) e^{-iy_{n+1}x_{n+1}}\, dx_{n+1} \bigg) e^{-iyx}\, dx \bigg\|^2_{L_{\mathbf{p}_\varepsilon}(\mathbb{R}^n)} dy_{n+1} \\
&\le \int_{-\infty}^{\infty} \bigg(\|\dots\| \bigg\| \int_{\infty}^{\infty} f(\cdot, x_{n+1}) e^{-iy_{n+1}x_{n+1}}\, dx_{n+1} \bigg\|_{L_{p'_{\varepsilon_n}}} \|_{L_{p'_{\varepsilon_{n-1}}}} \dots \|_{L_{p'_{\varepsilon_1}}} \bigg)^{2/p'_{\varepsilon_1}} dy_{n+1}.
\end{aligned}$$

Since $p'_{\varepsilon_j} \le 2$, $j = 1, \dots, n$, from Minkowski's inequality we obtain inequality (1). If $\varepsilon_{n+1} = 1$, i.e., $p_{\varepsilon_{n+1}} = \infty$, then, first, applying Minkowski's inequality and next, the induction assumption, we obtain inequality (1).

Thus, using the interpolation properties of the spaces $L_{\mathbf{p}}$ (Theorem 1 and Lemma 1), we obtain the required assertion. □

Theorem 3. *Suppose that* $* = (n, n-1, \dots, 1)$, $\mathbf{1} < \mathbf{p} < \mathbf{2} < \mathbf{q} < \infty$, $\mathbf{0} < \mathbf{s}, \mathbf{t} \le \infty$, *and*

$$\frac{\mathbf{1}}{\mathbf{r}} = \frac{\mathbf{1}}{\mathbf{p}} - \frac{\mathbf{1}}{\mathbf{q}}, \qquad \frac{\mathbf{1}}{\mathbf{h}} = \left(\frac{\mathbf{1}}{\mathbf{s}} - \frac{\mathbf{1}}{\mathbf{t}} \right)_+;$$

here $(x)_+ = \max(0, x)$. *Then* $L^*_{\mathbf{rh}} \hookrightarrow M(L_{\mathbf{pt}} \to L_{\mathbf{qs}})$.

Proof. Using Theorem 2, Hölder's inequality, and the properties of the spaces $L^*_{\mathbf{pt}}$; we obtain

$$\|F^{-1}\varphi Ff\|_{L_{\mathbf{qs}}} \le c_1 \|\varphi Ff\|_{L^*_{\mathbf{q's}}} \le c_1 \|\varphi\|_{L^*_{\mathbf{rh}}} \|Ff\|_{L^*_{\mathbf{p't}}} \le c_2 \|\varphi\|_{L^*_{\mathbf{rh}}} \|f\|_{L_{\mathbf{pt}}};$$

this proves the assertion.

In the isotropic case, for $\mathbf{p} = (p, \dots, p) = \mathbf{t}$, $\mathbf{q} = (q, \dots, q) = \mathbf{s}$, $1 < p < 2 < q \le \infty$, $1/r = 1/p - 1/q$, it follows from Theorem 3 that

$$\|\varphi\|_{M(L_p \to L_q)} \le \sup_{t_j > 0} t_1^{1/r} \dots t_n^{1/r} \varphi^{*_1 \cdots *_n}(t_1, \dots, t_n) = \|\varphi\|_{L_{(r,\dots,r)(\infty,\dots,\infty)}(\mathbb{R}^n)}. \tag{2}$$

This estimate is sharper than the assertion of Hörmander's Theorem [11]: for $1 < p < 2 \le q \le \infty$, $1/r = 1/p - 1/q$,

$$\|\varphi\|_{M(L_p \to L_q)} \le c\|\varphi\|_{L_{r\infty}(\mathbb{R}^n)}.$$

Indeed, for the simplicity of the argument, let $n=2$. Then

$$\begin{aligned}\|\varphi\|_{L_{(r,r)(\infty,\infty)}} &= \sup_{t_1>0,\,t_2>0}(t_1t_2)^{1/r}\varphi^{*_1*_2}(t_1,t_2)\\ &\le \sup_{t_1>0,\,t_2>0}\frac{1}{{t_1}^{1-1/r}}\frac{1}{{t_2}^{1-1/r}}\int_0^{t_2}\int_0^{t_1}\varphi^{*_1*_2}(s_1,s_2)\,ds_1\,ds_2\\ &\le \sup_{t_1>0,\,t_2>0}\sup_{|e_2|=t_2}\frac{1}{|e_2|}\int_{e_2}\sup_{|e_1|=t_1}\frac{1}{|e_1|}\int_{e_1}|\varphi(x_1,x_2)|\,dx_1\,dx_2.\end{aligned}$$

It follows from the definition of a supremum that for arbitrary fixed $t_1>0\quad t_2>0$ there exists families of measurable sets $e_2(t_2)$, $\{e_1(t_1,x_2)\}_{x_2\in e_2(t_2)}$ such that $|e_2(t_2)|=t_2\quad |e_1(t_1,x_2)|=t_1$ and

$$\begin{aligned}\|\varphi\|_{L_{(r,r)(\infty,\infty)}} &\le 4\sup_{t_1>0,\,t_2>0}\frac{1}{|e_2(t_2)|^{1-1/r}}\int_{e_2(t_2)}\frac{1}{|e_1(t_1,x_2)|^{1-1/r}}\int_{e_1(t_1,x_2)}|\varphi(x_1,x_2)|\,dx_1\,dx_2\\ &= 4\sup_{t_1>0,\,t_2>0}\frac{1}{(t_1t_2)^{1-1/r}}\int_{e_2(t_2)}\int_{e_1(t_1,x_2)}|\varphi(x_1,x_2)|\,dx_1\,dx_2\\ &\le 4\sup_{|e|>0}\frac{1}{|e|^{1-1/r}}\int_e|\varphi(x)|\,dx\sim\|\varphi\|_{L_{r\infty}}.\end{aligned}$$

The following example shows that the spaces $L_{r\infty}(\mathbb{R}^2)$ and $L_{(r,r)(\infty,\infty)}(\mathbb{R}^2)$ are different. □

Example 1. Let $1<r<\infty$. Consider the function

$$\varphi(x_1,x_2)=\big(|x_1|\,|x_2|\big)^{-1/r}.$$

Obviously, $\sup_{t_1>0,\,t_2>0}t_1^{1/r}t_2^{1/r}\varphi^{*_1*_2}(t_1,t_2)<\infty$. Let us show that $\|\varphi\|_{L_{r\infty}}=\sup_{t>0}t^{1/r}\varphi^*(t)=\infty$. Indeed, for $\varepsilon>0$ we define the set $e_\varepsilon=\{(x_1,x_2)\colon\ \varepsilon\le x_1\le 1,\ \ 0\le x_2\le\varepsilon/x_1\}$. Then

$$\frac{1}{|e_\varepsilon|^{1-1/r}}\int_{e_\varepsilon}\varphi(x_1,x_2)\,dx_1\,dx_2\sim|\ln\varepsilon|^{1/r}\to\infty,\qquad \varepsilon\to 0,$$

and hence

$$\|g\|_{L_{r\infty}}\sim\sup_{|e|>0}\frac{1}{|e|^{1-1/r}}\left|\int_e\varphi(x)\,dx\right|=\infty.$$

A function f is said to be *monotone* (*nonincreasing*) *in the extended sense* in $\mathbb{R}^n$ if for any $x\in\mathbb{R}^n$ and $x_j\ne 0$, $j=1,\dots,n$, the following inequality holds:

$$|f(x)|\le\frac{B}{|x_1|\dots|x_n|}\left|\int_{[0,x]}f(y)\,dy\right|;$$

here $[0,x]=\{y\in\mathbb{R}^n\colon\ 0\le y_j\operatorname{sign}x_j\le|x^j|,\ j=1,\dots,n\}$.

Corollary 1. *Suppose that* $1<p<2<q<\infty$, $1\le\tau\le s\le\infty$, *and* φ *is a definite* (*positive or negative*) *monotone* (*in the extended sense*) *function. Then for* $\varphi\in M(L_{p\tau}\to L_{qs})$, *it is necessary and sufficient that*

$$\sup_{Q\in E_0}\frac{1}{|Q|^{1/p'+1/q}}\int_Q|\varphi(x)|\,dx<\infty,$$

where E_0 *is the set of all parallelepipeds with faces parallel to the Cartesian planes.*

Remark. This assertion was stated in the author's paper [12] (Corollary 1). The proof given there is valid only for the one-dimensional case ($n=1$).

Proof. Suppose that $1/r = 1/p - 1/q$. It follows from Theorem 1 of [12] that for a function φ of fixed sign from $M(L_{p\tau} \to L_{qs})$ the following estimate is valid:

$$\sup_{Q\in E_0} \frac{1}{|Q|^{1-1/r}} \left| \int_Q \varphi(x)\,dx \right| \le \|\varphi\|_{M(L_{p\tau}\to L_{qs})}.$$

Let us choose a parameter $r \in (1,\infty)$ and a function $f \in L_{(r\dots r)(\infty\dots\infty)}(\mathbb{R}^n)$. Then for pairs (p_0, q_0) and (p_1, q_1) such that $1/p_0 - 1/q_0 = 1/r$, from relation (2) we obtain

$$\|T_\varphi f\|_{L_{q_i}(\mathbb{R}^n)} \le c\|\varphi\|_{L_{(r\dots r)(\infty\dots\infty)}(\mathbb{R}^n)} \|f\|_{L_{p_i}(\mathbb{R}^n)}, \qquad i=0,1.$$

Applying the real interpolation method, we find that for $\tau \le s$, $1/p - 1/q = 1/r$

$$\|T_\varphi f\|_{L_{qs}(\mathbb{R}^n)} \le c\|\varphi\|_{L_{(r\dots r)(\infty\dots\infty)}(\mathbb{R}^n)} \|f\|_{L_{p\tau}(\mathbb{R}^n)},$$

, i.e.,

$$\begin{aligned}\|\varphi\|_{M(L_{p\tau}\to L_{qs})} &\le c \sup_{t_1>0,\dots,t_n>0} t_1^{1/r}\dots t_n^{1/r}\varphi^{*_1\dots *_n}(t_1\dots t_n)\\ &\le c\sup_{|e_n|>0}\frac{1}{|e_n|^{1-1/r}}\left(\int_{e_n}\dots\sup_{|e_1|>0}\left(\frac{1}{|e_1|^{1-1/r}}\int_{e_1}\varphi(x_1,\dots,x_n)\,dx_1\right)\dots dx_n\right).\end{aligned}$$

Since the function φ is monotone in the extended sense, we have

$$\begin{aligned}&\sup_{|e_n|>0}\frac{1}{|e_n|^{1-1/r}}\left(\int_{e_n}\dots\sup_{|e_1|>0}\left(\frac{1}{|e_1|^{1-1/r}}\int_{e_1}\varphi(x_1,\dots,x_n)\,dx_1\right)\dots dx_n\right)\\
&\quad\le B\sup_{|e_n|>0}\frac{1}{|e_n|^{1-1/r}}\\
&\quad\quad\times\left(\int_{e_n}\dots\sup_{|e_1|>0}\left(\frac{1}{|e_1|^{1-1/r}}\int_{e_1}\frac{1}{|x_1|\dots|x_n|}\int_{[0,x]}\varphi(y)\,dy\,dx_1\right)\dots dx_n\right)\\
&\quad\le B\sup_{Q\in E_0}\frac{1}{|Q|^{1/p'+1/q}}\int_Q\varphi(x)\,dx\sup_{|e_n|>0}\frac{1}{|e_n|^{1-1/r}}\\
&\quad\quad\times\left(\int_{e_n}\dots\sup_{|e_1|>0}\left(\frac{1}{|e_1|^{1-1/r}}\int_{e_1}\frac{1}{(|x_1|\dots|x_n|)^{\frac{1}{r}}}\,dx_1\right)\dots dx_n\right)\\
&\quad= Bc\sup_{Q\in E_0}\frac{1}{|Q|^{1/p'+1/q}}\left|\int_Q\varphi(x)\,dx\right|.\end{aligned}$$

The proof of the corollary is complete. □

3. The Hardy–Littlewood–Paley inequality for the series $f=\sum_{k=1}^\infty a_k\varphi_k(x)$ in the orthonormal system $\Phi=\{\varphi_k\}$, $|\varphi_k(x)|\le c$, in $L_2[0,1]^n$ is of the form

$$\|f\|^p_{L_p[0,1]^n} \le c\sum_{i=1}^\infty i^{p-2}(a_i^*)^p \qquad \text{for } 2<p<\infty, \tag{3}$$

$$\sum_{i=1}^\infty i^{p-2}(a_i^*)^p \le c\|f\|^p_{L_p[0,1]^n} \qquad \text{for } 1<p<2, \tag{4}$$

where $\{a_i^*\}_{i=1}^{\infty}$ is the nonincreasing rearrangement of the sequence $\{a_k\}_{k\in\mathbb{N}^n}$ of the Fourier coefficients of the function f in the system Φ.

In the one-dimensional case, inequalities (3) and (4) are the sharpest among similar inequalities (for the Hardy–Littlewood and Hausdorff–Young ineqialities, see [13], for Pitt's inequality, see [14], and for other inequalities, see [15, 16]). For multiple Fourier series, the situation is different; for example, for trigonometric systems there are also independent inequalities [15, 16]:

$$\|f\|^p_{L_p[0,1]^n} \le \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{p}{2}-1)(k_1+\dots+k_n)}\biggl(\sum_{m_1=2^{k_1}}^{2^{k_1+1}-1}\cdots\sum_{m_n=2^{k_n}}^{2^{k_n+1}-1}|a_{m_1\dots m_n}|^2\biggr)^{\frac{p}{2}} \tag{5}$$

for $2<p<\infty$ and $f\sim\sum_{m\in\mathbb{N}^n} a_m \exp\bigl(2\pi i\sum_{r=1}^n m_r x_r\bigr)$ and

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{p}{2}-1)(k_1+\dots+k_n)}\biggl(\sum_{m_1=2^{k_1}}^{2^{k_1+1}-1}\cdots\sum_{m_n=2^{k_n}}^{2^{k_n+1}-1}|a_{m_1\dots m_n}|^2\biggr)^{\frac{p}{2}} \le c\|f\|^p_{L_p[0,1]^n}, \tag{6}$$

for $1<p<2$. In this section, we present inequalities which yield (3)–(6).

Suppose that $\mathbf{m}=(m_1,\dots,m_n)$, $m_j\in\mathbb{N}$, and $\Phi_j=\{\varphi_k^j(x_j)\}_{k\in\mathbb{N}}$ are orthonormal systems of functions in $L_2[0,1]^{m_j}$ such that

$$\sup_{x_j\in[0,1]^{m_j},\,k\in\mathbb{N}} |\varphi_k^j(x_j)|\le c, \qquad j=1,2,\dots,n.$$

The system $\Psi=\{\psi_k(x)\}_{k\in\mathbb{N}^n}$, $x\in[0,1]^{\mathbf{m}}=[0,1]^{m_1}\times\dots\times[0,1]^{m_n}$,

$$\psi_k(x)=\varphi_k^1(x_1)\dots\varphi_k^n(x_n), \tag{7}$$

is an orthonormal system of bounded functions in $L_2[0,1]^{\mathbf{m}}$.

Theorem 4. *Suppose that* Ψ *is an* orthonormal system of functions (7), $f\sim\sum_{k\in\mathbb{N}^n} a_k\psi_k(x)$, $*=(j_1,\dots,j_n)$ *is a rearrangement of the sequence* $(1,2,\dots,n)$, *and* $*'=(j_n,j_{n-1},\dots,j_1)$. *Then*

$$\|f\|_{L^*_{\mathbf{pq}}[0,1]^{\mathbf{m}}}\le c\|a\|_{l^{*'}_{\mathbf{p'q}}},$$

for $\mathbf{2}<\mathbf{p}=(p_1,\dots,p_n)<\infty$, $\mathbf{0}<\mathbf{q}=(q_1,\dots,q_n)\le\infty$, *and* $\mathbf{p'}=\mathbf{p}/(\mathbf{p}-\mathbf{1})$ *and*

$$\|a\|_{l^{*'}_{\mathbf{p'q}}}\le c\|f\|_{L^*_{\mathbf{pq}}[0,1]^{\mathbf{m}}},$$

for $\mathbf{1}<\mathbf{p}=(p_1,\dots,p_n)<\mathbf{2}$, $\mathbf{0}<\mathbf{q}=(q_1,\dots,q_n)\le\infty$, *and* $\mathbf{p'}=\mathbf{p}/(\mathbf{p}-\mathbf{1})$.

These inequalities for $*=(1,2,\dots,n)$ and $\mathbf{m}=(1,1,\dots,1)$ were proved in [10]. Theorem 4 is proved by using the interpolation Theorem 1 just as was done in the proof of Theorem 2 (see also Theorem 7 in [10]).

Note that the order of nonincreasing rearrangements in Theorems 2 and 4 is essential. Thus, inequality

$$\|a\|_{l^*_{\mathbf{p'q}}}\le c\|f\|_{L^*_{\mathbf{pq}}[0,1]^{\mathbf{m}}} \tag{8}$$

is false (here, on the left-hand side of the inequality, we have $*$ instead of $*'$, as required). Indeed, suppose that

$$n=2,\quad *=(1,2),\quad 1<p_1<p_2<2,\ \ q_2<\infty,\quad \varphi(x)\sim\sum_{k=1}^{\infty}\frac{1}{k^{1/p_2'}}\exp(2\pi ikx).$$

Consider the function $f(x,y)=\varphi(x+y)$; its Fourier coefficients are

$$a_{k_1k_2}=\begin{cases} k^{-1/p_2'}, & k_1=k_2=k>0,\\ 0 & \text{otherwise,}\end{cases}$$

where $k_1,k_2\in\mathbb{N}$. Then the common term of the nonincreasing rearrangement $\{a^{*_1*_2}_{k_1k_2}\}$ is of the form

$$a^{*_1*_2}_{k_1k_2}=\begin{cases} k_2^{-1/p_2'}, & k_1=1,\\ 0, & k_1>1.\end{cases}$$

It follows that $\|a\|_{l^*_{\mathbf{p}'\mathbf{q}}}=\big\|\{1/k^{1/p_2'}\}\big\|_{l_{p_2'q_2}}=\infty$. On the other hand, $f^{*_1}(t,y)=\varphi^*(t)$ and

$$\|f\|_{L^*_{\mathbf{pq}}}=\|\varphi\|_{L_{p_1q_1}}\sim\big\|\{1/k^{1/p_2'}\}\big\|_{l_{p_1'q_1}}<\infty;$$

therefore, inequality (8) fails.

If $\mathbf{m}=(1,\dots,1)$, $\mathbf{p}=(p,\dots,p)=\mathbf{q}$, then from Theorem 4 we obtain the inequalities

$$\|f\|^p_{L_p[0,1]^n}\le c\sum_{k_1=1}^\infty\cdots\sum_{k_n=1}^\infty k_1^{p-2}\dots k_n^{p-2}\big(a^{*_{j_1}*_{j_2}\cdots *_{j_n}}_{k_1k_2\dots k_n}\big)^p \qquad \text{for } 2<p<\infty, \tag{9}$$

$$\sum_{k_1=1}^\infty\cdots\sum_{k_n=1}^\infty k_1^{p-2}\dots k_n^{p-2}\big(a^{*_{j_1}*_{j_2}\cdots *_{j_n}}_{k_1k_2\dots k_n}\big)^p\le c\|f\|^p_{L_p[0,1]^n} \qquad \text{for } 1<p<2\,. \tag{10}$$

Let us show that these inequalities are sharper in the case of a system of the form (7) than in the cases of (3) and (4). Suppose that $A_r=\{\mathbf{m}\in\mathbb{N}^n\colon\ m_1\cdot m_2\dots m_n=r\}$, $r\in\mathbb{N}$, and

$$\sum_{k_1=1}^\infty\cdots\sum_{k_n=1}^\infty k_1^{p-2}\dots k_n^{p-2}(a^{*_1\cdots *_n}_{k_1\dots k_n})^p=\sum_{r=1}^\infty r^{p-2}\sum_{\mathbf{m}\in A_r}(a^{*_1\cdots *_n}_{m_1\dots m_n})^p. \tag{11}$$

On the other hand,

$$\sum_{r=1}^\infty r^{p-2}(a_r^*)^p=\sum_{r=1}^\infty\sum_{\mathbf{m}\in A_r}\beta^{p-2}_{\mathbf{m}}(a^{*_1\cdots *_n}_{m_1\dots m_n})^p;$$

here the sequence $\{\beta^{p-2}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^n}$ is the corresponding rearrangement of the sequence $\{r^{p-2}\}_{r\in\mathbb{N}}$.

Since $a^{*_1\cdots *_n}_{k_1\dots k_n}\ge a^{*_1\cdots *_n}_{m_1\dots m_n}$ for all $1\le k_j\le m_j$, $j=1,2,\dots,n$, we see that that there exist at least, $m_1\cdot m_2\dots m_n$ terms of the sequence $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^n}$ which are not less than the number $a^{*_1\cdots *_n}_{m_1\dots m_n}$ in absolute value, and hence $\beta_{m_1,\dots,m_n}>m_1\cdot m_2\dots m_n$. Therefore, we have

$$\sum_{r=1}^\infty r^{p-2}(a_r^*)^p\ge\sum_{r=1}^\infty r^{p-2}\sum_{\mathbf{m}\in A_r}(a^{*_1\cdots *_n}_{m_1\dots m_n})^p \qquad \text{for}\quad 2<p;$$

$$\sum_{r=1}^\infty r^{p-2}(a_r^*)^p\le\sum_{r=1}^\infty r^{p-2}\sum_{\mathbf{m}\in A_r}(a^{*_1\cdots *_n}_{m_1\dots m_n})^p \qquad \text{for}\quad p<2.$$

Taking relation (11) into account, we see that inequalities (9), (10) yield (3), (4), respectively.

The following example shows that inequalities (9) and (3) are not equivalent.

Example 2. Suppose that $p > 3$, $0 < \varepsilon < p-3$, $p' = p/(p-1)$, and

$$a_{k_1k_2} = (k_1k_2)^{-1/p'}(\ln(k_1k_2+1))^{-(2+\varepsilon)/p}, \qquad k_1, k_2 \in \mathbb{N}.$$

Obviously,

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} k_1^{p-2}k_2^{p-2}(a_{k_1k_2}^{*_1*_2})^p < \infty.$$

Suppose that $A_r = \{\mathbf{m} \in \mathbb{N}^2\colon\ m_1m_2 = r\}$, $r \in \mathbb{N}$. Just as above, we can write

$$\sum_{r=1}^{\infty} r^{p-2}(a_r^*)^p = \sum_{r=1}^{\infty}\sum_{\mathbf{m}\in A_r} \beta_{m_1m_2}^{p-2}(a_{m_1,m_2}^{*_1*_2})^p.$$

Since $(a_{k_1k_2}^{*_1*_2}) > (a_{m_1m_2}^{*_1*_2})$ for $k_1{\cdot}k_2 < m_1{\cdot}m_2$, it follows that $\beta_{m_1m_2} > \sum_{s=1}^{r-1}|A_s|$ for $m_1{\cdot}m_2 > r$; therefore,

$$\sum_{r=1}^{\infty} r^{p-2}(a_r^*)^p \ge \sum_{r=1}^{\infty}\biggl(\sum_{s=1}^{r-1}|A_s|\biggr)^{p-2}|A_r|\frac{1}{r^{p-1}(\ln r+1)^{2+\varepsilon}}.$$

The number of elements of $|A_r|$ in the set A_r is not less than the number of different divisors of the number r. It is well known that the sum of all different divisors of numbers from 1 to r is asymptotically equal to $r\ln r$. Therefore,

$$\sum_{r=1}^{\infty} r^{p-2}(a_r^*)^p \ge c\sum_{r=1}^{\infty}\frac{|A_r|}{r(\ln r+1)^{4-p+\varepsilon}} \ge c\sum_{r=1}^{\infty}\frac{1}{r(\ln r+1)^{4-p+\varepsilon}} = \infty.$$

Let us now show that inequality (9) yields (5).

Let $2 < p < \infty$. Note that for any h: $1 \le h \le \infty$

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} k_1^{p-2}\dots k_n^{p-2}(a_{k_1\dots k_n}^{*_1\cdots *_n})^p$$
$$\sim \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{1}{p'}-\frac{1}{h})p(k_1+\cdots+k_n)}\biggl(\sum_{m_1=2^{k_1}}^{2^{k_1+1}-1}\cdots\sum_{m_n=2^{k_n}}^{2^{k_n+1}-1}(a_{m_1\dots m_n}^{*_1\cdots *_n})^h\biggr)^{p/h}.$$

Taking $h = 2$, we obtain

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} k_1^{p-2}\dots k_n^{p-2}(a_{k_1\dots k_n}^{*_1\cdots *_n})^p$$
$$\sim \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{1}{p'}-\frac{1}{2})p(k_1+\cdots+k_n)}\biggl(\sum_{m_1=2^{k_1}}^{2^{k_1+1}-1}\cdots\sum_{m_n=2^{k_n}}^{2^{k_n+1}-1}(a_{m_1\dots m_n}^{*_1\cdots *_n})^2\biggr)^{p/2}$$
$$\le \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{1}{2}-\frac{1}{p})p(k_1+\cdots+k_n)}\biggl(\sum_{m_1=2^{k_1}}^{\infty}\cdots\sum_{m_n=2^{k_n}}^{\infty}(a_{m_1\dots m_n}^{*_1\cdots *_n})^2\biggr)^{p/2}$$
$$\le \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{p}{2}-1)(k_1+\cdots+k_n)}\biggl(\sum_{m_1=2^{k_1}}^{\infty}\cdots\sum_{m_n=2^{k_n}}^{\infty}|a_{m_1\dots m_n}|^2\biggr)^{p/2}$$
$$= \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2^{(\frac{p}{2}-1)(k_1+\cdots+k_n)}\biggl(\sum_{\nu_1=k_1}^{\infty}\cdots\sum_{\nu_n=k_n}^{\infty}\sum_{m_1=2^{\nu_1}}^{2^{\nu_1+1}-1}\cdots\sum_{m_n=2^{\nu_n}}^{2^{\nu_n+1}-1}|a_{m_1\dots m_n}|^2\biggr)^{p/2}.$$

Further, applying one of the generalizations of Hardy's inequality (see [17]), the last expression can be estimated via

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}2^{(\frac{p}{2}-1)(k_1+\cdots+k_n)}\bigg(\sum_{m_1=2^{k_1}}^{2^{k_1+1}-1}\cdots\sum_{m_n=2^{k_n}}^{2^{k_n+1}-1}|a_{m_1\dots m_n}|^2\bigg)^{p/2},$$

as was required.

For trigonometric series, Bugrov obtained the following estimate (see [18]): for

$$1<p_n\le p_{n-1}\le\cdots\le p_1<2,\qquad f\sim\sum_{m\in\mathbb{Z}^n}a_m\exp(2\pi i\sum_{r=1}^{n}m_rx_r),\quad f\in L_{\mathbf{p}}[0,1]^n,$$

the following inequality is valid:

$$\bigg(\sum_{m_1\in\mathbb{Z}}(|m_1|+1)^{p_1-2}\bigg(\dots\bigg(\sum_{m_n\in\mathbb{N}}(|m_n|+1)^{p_n-2}|a_{m_1\dots m_n}|^{p_n}\bigg)^{p_{n-1}/p_n}\dots\bigg)^{p_1/p_2}\bigg)^{1/p_1}$$
$$\le c\bigg(\int_0^1\bigg(\dots\bigg(\int_0^1|f(x_1,\dots,x_n)|^{p_1}\,dx_1\bigg)^{p_2/p_1}\dots\bigg)^{p_n/p_{n-1}}dx_n\bigg)^{1/p_n}. \tag{12}$$

Let us show that Theorem 4 yields (12). Without loss of generality, we can assume that $n=2$. Taking into account the fact that $p_j<2$ and, therefore, the sequences $\{(|m_j|+1)^{p_j-2}\}$, $j=1,2$, are symmetrically decreasing, we obtain

$$I=\bigg(\sum_{m_1\in\mathbb{Z}}(|m_1|+1)^{p_1-2}\bigg(\bigg(\sum_{m_2\in\mathbb{Z}}(|m_2|+1)^{p_2-2}|a_{m_1m_2}|^{p_2}\bigg)^{\frac{1}{p_2}}\bigg)^{p_1}\bigg)^{\frac{1}{p_1}}$$
$$\le 4\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\bigg(\bigg[\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}|a^{*_2}_{m_1k_2}|^{p_2}\bigg)^{\frac{1}{p_2}}\bigg]^{*_1}_{k_1}\bigg)^{p_1}\bigg)^{\frac{1}{p_1}}$$
$$\le 4\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\sup_{\substack{|e|=k_1\\ e\subset\mathbb{Z}}}\bigg(\frac{1}{|e|}\sum_{m_1\in e}\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}|a^{*_2}_{m_1k_2}|^{p_2}\bigg)^{\frac{2}{p_2}}\bigg)^{\frac{p_1}{2}}\bigg)^{\frac{1}{p_1}}$$
$$\le 4\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}\sup_{\substack{|e|=k_1\\ e\subset\mathbb{Z}}}\bigg(\frac{1}{|e|}\sum_{m_1\in e}|a^{*_2}_{m_1k_2}|^{2}\bigg)^{\frac{p_2}{2}}\bigg)^{\frac{p_1}{p_2}}\bigg)^{\frac{1}{p_1}}$$
$$=4\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}\bigg(\frac{1}{k_1}\sum_{r_1=1}^{k_1}(a^{*_2*_1}_{r_1k_2})^2\bigg)^{\frac{p_2}{2}}\bigg)^{\frac{p_1}{p_2}}\bigg)^{\frac{1}{p_1}}.$$

Since

$$\bigg(\frac{1}{k}\sum_{r=1}^{k}(b_r^*)^2\bigg)^{\frac{1}{2}}\le c\frac{1}{k^{\frac{1}{2}}}\sum_{r=1}^{k}r^{-\frac{1}{2}}b_r^*,$$

where c is independent of k and of the sequence $\{b_r\}$, it follows that

$$I\le c_1\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}\bigg(\frac{1}{k_1^{\frac{1}{2}}}\sum_{r=1}^{k_1}r_1^{-\frac{1}{2}}a^{*_2*_1}_{r_1k_2}\bigg)^{p_2}\bigg)^{\frac{p_1}{p_2}}\bigg)^{\frac{1}{p_1}}$$
$$\le c_1\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\bigg(\frac{1}{k_1^{\frac{1}{2}}}\sum_{r_1=1}^{k_1}r_1^{-\frac{1}{2}}\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}(a^{*_2*_1}_{r_1k_2})^{p_2}\bigg)^{\frac{1}{p_2}}\bigg)^{p_1}\bigg)^{\frac{1}{p_1}}$$
$$\le c_2\bigg(\sum_{k_1=1}^{\infty}k_1^{p_1-2}\bigg(\sum_{k_2=1}^{\infty}k_2^{p_2-2}(a^{*_2*_1}_{k_1k_2})^{p_2}\bigg)^{\frac{p_1}{p_2}}\bigg)^{\frac{1}{p_1}}.$$

Now we use Theorem 4, obtaining

$$I \le c_3\left(\int_0^1\left(\int_0^1 \left(f^{*_1,*_2}(t_1,t_2)\right)^{p_2} dt_2\right)^{\frac{p_1}{p_2}} dt_1\right)^{\frac{1}{p_1}} = c_3\left(\int_0^1\left(\int_0^1 \left(f^{*_1}(t_1,x_2)\right)^{p_2} dx_2\right)^{\frac{p_1}{p_2}} dt_1\right)^{\frac{1}{p_1}}$$

$$\le c_3\left(\int_0^1\left(\int_0^1 |f(x_1,x_2)|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{1}{p_2}}.$$

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