INTERPOLATION METHODS FOR STOCHASTIC PROCESSES SPACES

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ABSTRACT. In this paper the scales of classes of stochastic processes are introduced. New interpolation theorems and boundedness of some transforms of stochastic processes are proved. Interpolation method for generously-monotonous processes is entered. Conditions and statements of interpolation theorems concern the fixed stochastic process, which differs from the classical results.

1. Introduction

Interpolation methods of functional spaces are one of the basic tools to get inequalities in parametrical spaces. These methods are widely applied in the theory of stochastic processes (see [6] - [5] and other).

In this paper classes of stochastic processes are considered, which, in some sense, are analogues of the Net spaces which were investigated in [14], [13]. These classes are defined by a functional which depends on parametres connected with such important notions as the theory of random processes, the convergence of process, the law of great numbers, the martingale properties of process.

Assume that $(\Omega, \mathfrak{G}, P)$ is a complete probability space. A family $G = \{\mathfrak{G}_n\}_{n\geq 1}$ of σ - algebras $\mathfrak{G}_n$ such that $\mathfrak{G}_1 \subseteq ... \subseteq \mathfrak{G}_n \subseteq ... \subseteq \mathfrak{G}$ is called a filtration.

Let G be a filtration, a sequence $\{X_n\}_{n\geq 1}$ of random variables X_n be such that for any $n \geq 1$ X_n is a measurable function with respect to the σ-algebra $\mathfrak{G}_n$. Then we say that the set $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ is a stochastic process.

Let $F = \{\mathfrak{F}_n\}_{n\geq 1}$ be a system of sets satisfying the condition $\mathfrak{F}_1 \subseteq ... \subseteq \mathfrak{F}_n \subseteq ... \subseteq \mathfrak{G}$. We say that a stochastic process $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ is defined on a system $F = \{\mathfrak{F}_n\}_{n\geq 1}$ if $\mathfrak{F}_n \subset \mathfrak{G}_n,\ n \in \mathbb{N}$. For a stochastic process X, which is defined on a system $F = \{\mathfrak{F}_n\}_{n\geq 1}$, we define the sequence of numbers $\overline{X}(F) = \{\overline{X}_n(F)\}_n$, where

$$\overline{X}_k(F) = \sup_{A\in\mathfrak{F}_k, P(A)>0} \frac{1}{P(A)} \left| \int_A X_k P(d\omega) \right|.$$

We call this sequence a majorant of a process X on a system of sets F.

Let us give some examples of a choice of a system of sets $F = \{\mathfrak{F}_n\}_{n\geq 1}$: $\mathfrak{F}_n = \Omega$, in this case the sequence $\overline{X}(F) = \{\overline{X}_n(F)\}_n$ is a sequence of averages of a process $X = \{X_n\}$; for $\mathfrak{F}_n = \mathfrak{G}_{n-1}$ it is a majorant of sequence of conditional averages $M(X_n|G_{n-1})$; and for $\mathfrak{F}_n = \mathfrak{G}_n$ it is a majorant of a process $X = \{X_n\}$. The

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following cases are interesting: 1) $\mathfrak{F}_n = \mathfrak{G}_{n\wedge\tau}$, where $\tau = \tau(\omega)$ is the fixed stopping time, $\mathfrak{G}_{n\wedge\tau} = \{A \in \mathfrak{G} : A \cap \{\tau = n\} \in \mathfrak{G}_n\}$; 2) $\mathfrak{F}_n = \mathfrak{G}_{n\wedge\tau_n}$, where $\tau_n = \tau_n(\omega), n \in \mathbb{N}$ is the sequence of the stopping times.

We consider the classes of stochastic processes defined on F, which characterize the speed of convergence of sequence $\left\{\frac{\bar{X}_n(F)}{n}\right\}_n$ to zero.

By $N_{p,q}(F)$, $0 < p < \infty$, $0 < q \leq \infty$ we denote the set of all stochastic processes X, defined on F for which

$$\|X\|_{N_{p,q}(F)} = \left(\sum_{k=1}^{\infty} k^{-1-\frac{q}{p}}\overline{X}_k^q\right)^{\frac{1}{q}} < \infty \tag{1}$$

if $0 < q < \infty$ and

$$\|X\|_{N_{p,\infty}(F)} = \sup_k k^{-\frac{1}{p}}\overline{X}_k < \infty \tag{2}$$

if $q = \infty$.

Let us denote

$$N_p^{\alpha,q}(F) = \left\{X = (X_n, F_n)_{n\geq 1} : \left(\sum_{k=0}^{\infty}(2^{\alpha k}\overline{\Delta X_k})^q\right)^{\frac{1}{q}} < \infty\right\}$$

for $0 < q < \infty$, and

$$N_p^{\alpha,\infty}(F) = \{X = (X_n, F_n)_{n\geq 1} : \sup_k 2^{\alpha k}\overline{\Delta X_k} < \infty\},$$

for $q = \infty$, where

$$\overline{\Delta X_k} = \sup_{A\in\mathfrak{F}, P(A)>0} \frac{1}{(P(A))^{1-\frac{1}{p}}} \left|\int_A (X_{2^k} - X_{2^{k-1}})P(d\omega)\right|.$$

We consider that $X_{\frac{1}{2}}(\omega) \equiv 0$.

The $N_p^{\alpha,q}(F)$ spaces are spaces of converging stochastic processes, where parametres α, q and p characterize the speed and the metrics, in which a given process converges.

In this paper we prove a Marcinkiewicz-type interpolation theorem for the introduced space. An interpolation method, essentially related to the properties of the Markov stopping times, is introduced. In the last paragraph the given interpolation method is applied to Besov type space with variable approximation properties.

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \leqslant cB$ (or $cA \geqslant B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B. Notation $A \asymp B$ means that $A \lesssim B$ and $A \gtrsim B$.

2. Properties of the spaces $N_{p,q}(F)$.

We say ([15]) that stochastic process $(X_n, \mathfrak{G}_n)_{n\geq 1}$, is a martingale if for every $n \in \mathbb{N}$ the following conditions hold: 1) $E|X_n| < \infty$; 2) $E(X_{n+1}|\mathfrak{G}_n) = X_n$ (a.p.). If instead of property 2) it is assumed that $E(X_{n+1}|\mathfrak{G}_n) \geq X_n$ $(E(X_{n+1}|\mathfrak{G}_n) \leq X_n)$, then we say that a process $X = (X_n, \mathfrak{G}_n)_{n=1}^{\infty}$ is a submartingale (supermartingale).

Definition 1. *Let $F = \{\mathfrak{F}_n)\}_n$ be a fixed system of sets, $X = (X_n, \mathfrak{G}_n)_n$ be a stochastic process defined on F. We say that a process X belongs to the class $W(F)$ if there exists a constant c such that for every $k \leq m$ and for every $A \in \mathfrak{F}_k$*

$$\left|\int_A X_k P(d\omega)\right| \leq C \left|\int_A X_m P(d\omega)\right|.$$

This inequality implies that $\overline{X}_k(F) \leq c\overline{X}_m(F)$ for every $k \leq m$. The class $W(F)$ contains martingales, nonnegative submartingales, and non-positive supermartingales. The property of a process which is determined by its belonging to the class $W(F)$ we call the generalized monotonicity.

Lemma 1. *Let $X \in W(F)$. Then*

1) *for $0 < q \leq q_1 \leq \infty$,*

$$\|X\|_{N_{p,q_1}(F)} \leq c_{p,q,q_1} \|X\|_{N_{p,q}(F)},$$

2) *for $0 < p_1 < p < \infty$, $0 < q, q_1 \leq \infty$,*

$$\|X\|_{N_{p_1,q_1}(F)} \leq c_{p,q,p_1,q_1} \|X\|_{N_{p,q}(F)}$$

where $c_{p,q,q_1}, c_{p,q,p_1,q_1} > 0$ depend only on the indicated parameters.

Remark. Here and in the sequel constants c, c_1 etc. may be different in different formulas.

Proof. Let $\varepsilon > 0$. By Minkowski's inequality and by the generalized monotonicity of a process $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ we get

$$\|X\|_{N_{p,q_1}(F)} = \left(\sum_{k=1}^{\infty} k^{\varepsilon q_1 - 1} k^{-\varepsilon q_1 - \frac{q_1}{p}} \overline{X}_k^{q_1}\right)^{\frac{1}{q_1}} \lesssim$$

$$\lesssim \left(\sum_{k=1}^{\infty} k^{\varepsilon q_1 - 1}\left(\sum_{r=k}^{\infty} r^{-q(\varepsilon+\frac{1}{p})-1}\overline{X}_r^q\right)^{\frac{q_1}{q}}\right)^{\frac{1}{q_1}} \lesssim$$

$$\lesssim \left(\sum_{r=1}^{\infty} r^{-q(\varepsilon+\frac{1}{p})-1}\overline{X}_r^q\left(\sum_{k=1}^{r} k^{\varepsilon q_1 - 1}\right)^{\frac{q}{q_1}}\right)^{\frac{1}{q}} \lesssim$$

$$\lesssim \left(\sum_{r=1}^{\infty} r^{-\frac{q}{p}-1}\overline{X}_r^q\right)^{\frac{1}{q}} = \|X\|_{N_{p,q}(F)}.$$

To prove the second statement it is enough to show that $\|X\|_{N_{p_1,q_1}(F)} \leq \|X\|_{N_{p,\infty}(F)}$ and apply the first statement. Since $p_1 < p$, we have

$$\|X\|_{N_{p_1,q_1}(F)} = \left(\sum_{k=1}^{\infty} k^{-\frac{q_1}{p_1}-1}\overline{X}_k^{q_1}\right)^{\frac{1}{q_1}} \leq$$

$$\leq \sup_k k^{-\frac{1}{p}}\overline{X}_k\left(\sum_{k=1}^{\infty} k^{\frac{q_1}{p}-\frac{q_1}{p_1}-1}\right)^{\frac{1}{q_1}} = c\|X\|_{N_{p,\infty}(F)}.$$

Lemma 2. *Let $0 < p < \infty$, $a > 1$. If $X \in W(F)$, then for $0 < q < \infty$*

$$\|X\|_{N_{p,q}(F)} \asymp \left(\sum_{k=0}^{\infty}\left(a^{-\frac{k}{p}}\overline{X}_{a^k}\right)^q\right)^{\frac{1}{q}}, \tag{3}$$

and for $q = \infty$

$$\|X\|_{N_{p,\infty}(F)} \asymp \sup_{k\in\mathbb{N}} a^{-\frac{k}{p}}\overline{X}_{a^k}.$$

Here by $\overline{X}_{a^k}$ we mean $\overline{X}_{[a^k]}$, where $[a^k]$ is the integer part of the number a^k. Moreover, expressions X^α and $\sum_{k=\alpha}^{\beta} b_k$ will be understood as $X^{[\alpha]}$, $\sum_{k=[\alpha]}^{[\beta]} b_k$ respectively.

Proof. Using the generalized monotonicity of a process X, we have

$$\|X\|_{N_{p,q}(F)} = \left(\sum_{k=1}^{\infty} k^{-\frac{q}{p}-1}\overline{X}_k^q\right)^{\frac{1}{q}} = \left(\sum_{k=0}^{\infty}\sum_{l=a^k}^{a^{k+1}-1} l^{-\frac{q}{p}-1}\overline{X}_l^q\right)^{\frac{1}{q}} \gtrsim$$

$$\gtrsim \left(\sum_{k=0}^{\infty} a^{-\frac{kq}{p}}\overline{X}_{a^k}^q\sum_{l=a^k}^{a^{k+1}-1}\frac{1}{l}\right)^{\frac{1}{q}} \gtrsim \left(\sum_{k=0}^{\infty}\left(a^{-\frac{k}{p}}\overline{X}_{a^k}\right)^q\right)^{\frac{1}{q}}.$$

One can prove the reverse estimate in a similar way.

Lemma 3. (Hölder inequality). *Let* $0 < p_1, p_2, q < \infty, 0 < t, s_1, s_2 \le \infty$ *and* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{t} = \frac{1}{s_1} + \frac{1}{s_2}$. *If stochastic processes* $X = (X_n, \mathfrak{G}_n) \in N_{p_1,s_1}(F)$ *and* $Y = (Y_n, \mathfrak{G}_n) \in N_{p_2,s_2}(G)$, *then* $XY = (X_n Y_n, \mathfrak{G}_n) \in N_{q,t}(F)$ *and*

$$\|XY\|_{N_{q,t}(F)} \le \|X\|_{N_{p_1,s_1}(F)} \|Y\|_{N_{p_2,s_2}(G)}.$$

Proof. Since Y_n is measurable with respect to an algebra $\mathfrak{G}_n$, we have $\overline{X_k Y_k}(F) \le \overline{X_k}(F)\ \overline{Y_k}(G)$ and hence

$$\|XY\|_{N_{q,t}(F)} = \left(\sum_{k=1}^{\infty} (k^{-\frac{1}{p}} \overline{X_k Y_k})^t \frac{1}{k}\right)^{\frac{1}{t}} \le$$

$$\le \left(\sum_{k=1}^{\infty} (k^{-\frac{1}{p_1}} \overline{X_k}(F))^{s_1} \frac{1}{k}\right)^{\frac{1}{s_1}} \left(\sum_{k=1}^{\infty} (k^{-\frac{1}{p_2}} \overline{Y_k}(G))^{s_2} \frac{1}{k}\right)^{\frac{1}{s_2}} =$$

$$= \|X\|_{N_{p_1,s_1}(F)} \|Y\|_{N_{p_2,s_2}(G)}.$$

The lemma is proved.

We will need the following Hardy-type inequalities.

Lemma 4. *Let* $s \ge 1,\ \nu > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0$, *then for a nonnegative sequence* $a = \{a_k\}_k$ *the following inequalities hold:*

$$\left(\sum_{k=1}^{\infty} k^{-\alpha s - 1} \left(\sum_{l=1}^{(\gamma k)^\nu} l^{\beta-1} a_l\right)^s\right)^{1/s} \le \gamma^\alpha C_{\alpha,\beta,s,\nu} \left(\sum_{k=1}^{\infty} k^{(\beta - \frac{\alpha}{\nu})s - 1} a_k^s\right)^{1/s},$$

$$\left(\sum_{k=1}^{\infty} k^{\alpha s - 1} \left(\sum_{l=(\gamma k)^\nu}^{\infty} l^{\beta-1} a_l\right)^s\right)^{1/s} \le \gamma^{-\alpha} C_{\alpha,\beta,s,\nu} \left(\sum_{k=1}^{\infty} k^{(\beta + \frac{\alpha}{\nu})s - 1} a_k^s\right)^{1/s},$$

$$\left(\sum_{k=0}^{\infty} \left(2^{-\alpha k} \sum_{m=0}^{\gamma k} 2^{\beta m} a_m\right)^s\right)^{1/s} \le C_{\alpha,\beta,s,\gamma} \left(\sum_{k=1}^{\infty} \left(2^{(\beta - \frac{\alpha}{\nu})k} a_k\right)^s\right)^{1/s},$$

$$\left(\sum_{k=0}^{\infty} \left(2^{\alpha k} \sum_{m=\gamma k}^{\infty} 2^{\beta m} a_m\right)^s\right)^{1/s} \le C_{\alpha,\beta,s,\gamma} \left(\sum_{k=1}^{\infty} \left(2^{(\beta + \frac{\alpha}{\nu})k} a_k\right)^s\right)^{1/s}.$$

3. Interpolation theorem.

Let $\mathbf{T} = \{T_n\}_{n=1}^{\infty}$ be a transform that transforms a stochastic process X, which is defined on the system $F = \{\mathfrak{F}\}_{n=1}^{\infty}$, to the stochastic process $\mathbf{T}(X) = \{T_n(X), \Phi_n\}_{n=1}^{\infty}$, which is defined on the system $R = \{\mathfrak{R}\}_{n=1}^{\infty}$. We say that the transform T is quasilinear if there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ the following inequality holds almost surely

$$|T_n(X) - T_n(Y)| \le C |T_n(X - Y)|. \tag{4}$$

A random variable τ, which takes values in the set $(1, 2..., \infty)$, is called the Markov time of the filtration $G = \{\mathfrak{G}_n\}_{n\geq 1}$, if $\{\omega : \tau(\omega) = n\} \in \mathfrak{G}_n$ for any $n \in \mathbb{N}$. The Markov time τ, for which $\tau(\omega) < \infty$ (a.p.) [15], is called the stopping time.

Let $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ be a stochastic process, τ be the Markov time. By X^τ we denote the stopped process $X^\tau = (X_{n\wedge\tau}, \mathfrak{G}_n)$, where $X_{n\wedge\tau} = \sum_{m=1}^{n-1} X_m \chi_{\tau=m}(\omega) + X_n \chi_{\tau\geq n}(\omega)$ and $\chi_A(\omega)$ is the characteristic function of the set A.

It is known ([15]) that if a process $X = (X_n, \mathfrak{G}_n)_{n\geq 1}$ is a martingale (submartingale), then the process $X^\tau = (X_{n\wedge\tau}, \mathfrak{G}_n)_{n\geq 1}$ is also a martingale (submartingale).

Denote $X_n^*(\omega) = \max_{1\leq k\leq n} |X_k(\omega)|$ and $X^* = (X_n^*, \mathfrak{G}_n)_{n\geq 1}$.

The transforms X^τ and X^* of the stochastic process X are examples of quasilinear transforms.

Theorem 1. *Let* $0 < p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$, $0 < \theta < 1$, $1 \leq s \leq \infty$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $X \in W(F)$ *and* $T = \{T_n\}_{n=1}^\infty$ *be a quasilinear transform. If for any* $k \in \mathbb{N}$ *the following conditions hold*

$$\|T(X^k)\|_{N_{q_1,\infty}(R)} \leq M_1 \|X^k\|_{N_{p_1,1}(F)}, \tag{5}$$

$$\|T(X - X^k)\|_{N_{q_0,\infty}(R)} \leq M_0 \|X - X^k\|_{N_{p_0,1}(F)}, \tag{6}$$

then

$$\|T(X)\|_{N_{q,s}(R)} \leq C M_0^{1-\theta} M_1^\theta \|X\|_{N_{p,s}(F)}, \tag{7}$$

where $C > 0$ *depends only on* $p_0, p_1, q_0, q_1, \theta$.

Proof. Let $\gamma > 0$, $\nu > 0$. Using quasilinearity of the transform T and Minkowski's inequality, we have

$$\|T(X)\|_{N_{q,s}(R)} = \left(\sum_{k=1}^\infty k^{-1-\frac{s}{q}} \overline{T_k(X)}^s\right)^{\frac{1}{s}} \lesssim$$

$$\lesssim \left(\left(\sum_{k=1}^\infty k^{-1-\frac{s}{q}} \overline{T_k(X^{(\gamma k)^\nu})}^s\right)^{\frac{1}{s}} + \left(\sum_{k=1}^\infty k^{-1-\frac{s}{q}} \overline{T_k(X - X^{(\gamma k)^\nu})}^s\right)^{\frac{1}{s}}\right). \tag{8}$$

Denote $\lambda_k = (\gamma k)^\nu$ for any $k \in \mathbb{N}$. Using the definition of functional (2), conditions (5) and (6), we have

$$\overline{T_k(X^{\lambda_k})} \leq k^{\frac{1}{q_1}} M_1 \|X^{\lambda_k}\|_{N_{p_1,1}(F)}, \qquad \overline{T_k(X - X^{\lambda_k})} \leq k^{\frac{1}{q_0}} M_0 \|X - X^{\lambda_k}\|_{N_{p_0,1}(F)}. \tag{9}$$

By definitions of processes X^k and $X - X^k$, and taking into account the generalized monotonicity of the sequences $\{\overline{X_k}\}$ and $\{\overline{X_\infty - X_k}\}$, we have

$$\|X^{\lambda_k}\|_{N_{p_1,1}(F)} = \sum_{l=1}^{\lambda_k-1} l^{-1-\frac{1}{p_1}}\overline{X}_l + \overline{X}_{\lambda_k}\sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_1}} \lesssim$$

$$\lesssim \left(\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}}\overline{X_l} + \lambda_k^{\frac{1}{p_0}-\frac{1}{p_1}}\sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_1}}\overline{X_l}\right),$$

$$\|X - X^{\lambda_k}\|_{N_{p_0,1}(F)} = \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}}(\overline{X_l - X_{\lambda_k}}) \lesssim \sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}}\overline{X_l}. \tag{10}$$

Substituting estimates (9) and (10) in the first summand of the right-hand side of (8) and using Minkowski's inequality we have

$$\left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}}\overline{T_k(X^{\lambda_k})}^s\right)^{\frac{1}{s}} \lesssim$$

$$\lesssim M_1\left(\sum_{k=1}^{\infty} k^{-1-\frac{s}{q}+\frac{s}{q_1}}\left(\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}}\overline{X_l} + \lambda_k^{\frac{1}{p_0}-\frac{1}{p_1}}\sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}}\overline{X_l}\right)^s\right)^{\frac{1}{s}} \lesssim$$

$$\lesssim M_1\left(\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}}\overline{X_l}\right)^s\right)^{\frac{1}{s}} + \left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\lambda_k^{\frac{1}{p_0}-\frac{1}{p_1}}\sum_{l=\lambda_k}^{\infty} l^{-1-\frac{1}{p_0}}\overline{X_l}\right)^s\right)^{\frac{1}{s}}\right). \tag{11}$$

Let us estimate the first summand in the right-hand side of (11).

$$\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}}\overline{X_l}\right)^s\right)^{\frac{1}{s}} = \left(\sum_{k=1}^{\infty} k^{(\frac{1}{q_1}-\frac{1}{q})-1}\left(\sum_{l=1}^{(\gamma k)^\nu} l^{\frac{1}{p_1'}-1}\left(\frac{\overline{X_l}}{l}\right)\right)^s\right)^{\frac{1}{s}}.$$

By applying Lemma 4 and using equalities

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$\nu = \left(\frac{1}{q_0} - \frac{1}{q_1}\right) / \left(\frac{1}{p_0} - \frac{1}{p_1}\right)$$

we obtain

$$\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\sum_{l=1}^{\lambda_k} l^{-1-\frac{1}{p_1}}\overline{X_l}\right)^s\right)^{\frac{1}{s}} \lesssim \gamma^{\frac{1}{q}-\frac{1}{q_1}}\|X\|_{N_{p,s}(F)}. \tag{12}$$

By using Lemma 4 we estimate the second summand of the right-hand side of (11).

$$\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(k^{-\frac{1}{q}+\frac{1}{q_1}}\lambda_k^{\frac{1}{p_0}-\frac{1}{p_1}}\sum_{l=\lambda_k}^{\infty}l^{-1-\frac{1}{p_1}}\overline{X}_l\right)^s\right)^{\frac{1}{s}}=$$

$$=\gamma^{\frac{1}{q_0}-\frac{1}{q_1}}\left(\sum_{k=1}^{\infty}k^{s(\frac{1}{q_0}-\frac{1}{q})-1}\left(\sum_{l=(\gamma k)^{\nu}}^{\infty}l^{\frac{1}{p_0'}-1}\frac{\overline{X}_l}{l}\right)^s\right)^{\frac{1}{s}}\lesssim$$

$$\lesssim\gamma^{\frac{1}{q}-\frac{1}{q_1}}\|X\|_{N_{p,s}(F)}. \tag{13}$$

Substituting estimates (12) and (13) in (11), we get

$$\left(\sum_{k=1}^{\infty}k^{-1-\frac{s}{q}}\overline{T_k(X^{\lambda_k})}^s\right)^{\frac{1}{s}}\lesssim M_1\gamma^{\frac{1}{q}-\frac{1}{q_1}}\|X\|_{N_{p,s}(F)}. \tag{14}$$

Applying the second estimates of (9) and (10) to the second summand of the right-hand side of (8), we have

$$\left(\sum_{k=1}^{\infty}k^{-1-\frac{s}{q}}\overline{T_k(X-X^{\lambda_k})}^s\right)^{\frac{1}{s}}\lesssim$$

$$\lesssim M_0\left(\sum_{k=1}^{\infty}k^{(\frac{1}{q_0}-\frac{1}{q})s-1}\left(\sum_{l=\lambda_k}^{\infty}l^{\frac{1}{p_0'}-1}\frac{\overline{X}_l}{l}\right)^s\right)^{\frac{1}{s}}\lesssim M_0\gamma^{\frac{1}{q}-\frac{1}{q_0}}\|X\|_{N_{p,s}(F)}. \tag{15}$$

Choosing $\gamma=M_0^{\frac{q_0q_1}{q_1-q_0}}M_1^{\frac{q_0q_1}{q_0-q_1}}$ and substituting (14) and (15) in (8), we obtain (7). The theorem is proved.

The classical interpolation Marcinkievicz - Calderon theorem follows from this Theorem. Let f be a measurable function on (Ω,μ) and

$$m(\sigma,f)=\mu\{x:|f(x)|>\sigma\}$$

its distribution function. The function

$$f^*(t)=\inf\{\sigma:m(\sigma,f)\le t\}$$

is called the nonincreasing rearrangement of the function f.

Let $0<p<\infty$ and $0<q\le\infty$. The Lorentz space $L_{pq}(\Omega,\mu)$ is defined as the set of all measurable functions f such that

$$\|f\|_{L_{p,q}}=\left(\int_0^{\infty}(t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q}<\infty$$

for $0 < q < \infty$ and

$$\|f\|_{L_{p,\infty}} = \sup_t \; t^{1/p} f^*(t) < \infty$$

for $q = \infty$.

Corollary 1 (Marcinkievicz-Calderon theorem)**.** *Let* $0 < p_0 < p_1 < \infty, 0 < q_0 < q < q_1 \leq \infty$, $\theta \in (0,1), 1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$. *If* T *is a quasilinear map and*

$$T\colon L_{p_i,1}(D,\nu) \to L_{q_i,\infty}(\Omega,\mu) \quad \text{with the norm } M_i, \quad i = 0,1,$$

then

$$T\colon L_{p,\tau}(D,\nu) \to L_{q,\tau}(\Omega,\mu) \quad \text{and} \quad \|T\| \leq M_0^{1-\theta} M_1^{\theta}.$$

Proof. Let $f \geq 0$, $f \in L_{p,q}(\Omega,\mu)$ and $f^*(t)$ be the nonincreasing rearrangement of f. Let us define sets $\Omega_n = \{w \in \Omega : f(w) \leq f^*(1/n)\}$, $n \in \mathbb{N}$. The sequence $X = (X_n, \mathfrak{F}_n)$ is a stochastic process from class $W(F)$, where $X_n(w) = \min\{f(w), f^*(1/n)\}$, $\mathfrak{F}_n$ is the minimal $\sigma-$ algebra containing set Ω and system of all measurable subsets Ω_n. By the property of monotonicity of f^* we have

$$\|f\|_{L_{p,q}(\Omega,\mu)} \asymp \left(\sum_{k=1}^{\infty} \left(k^{-\frac{1}{p}} f^*(1/k) \right)^q \frac{1}{k} \right)^{1/q} = \left(\sum_{k=1}^{\infty} \left(k^{-\frac{1}{p}} \overline{X_k} \right)^q \frac{1}{k} \right)^{1/q} = \|X\|_{N_{p,q}(F)}.$$

Let X^n, $X - X^n$ be the stopping time and, respectively, the starting time of the process X corresponding to the time n. Then we also have

$$\|X_n\|_{L_{p,q}(\Omega,\mu)} \asymp \|X^n\|_{N_{p,q}(F)}, \tag{16}$$

$$\|f - X_n\|_{L_{p,q}(\Omega,\mu)} \asymp \|X - X^n\|_{N_{p,q}(F)}. \tag{17}$$

For a quasilinear operator T we will define a transform $\mathbf{T}$ which transforms a sequence $X = \{X_k\}_{k=1}^{\infty}$ to the sequence $\mathbf{T}X = \{TX_k\}_{k=1}^{\infty}$. Define the system of sets $\Phi = \{\Phi_k\}_{k=1}^{\infty}, \Phi_k = \{\Omega_m\}_{m=1}^{k}$. Then for the stopped sequence $X^n = \big(X_{\min(k,n)}, F_k\big)_{k\geq 1}$, we have

$$\begin{aligned}
\|TX^n\|_{N_{q_0,\infty}(F)} &= \sup_k k^{\frac{1}{q_0}} \sup_{A\in\Phi_k} \frac{1}{P(A)} \left| \int_A TX_{\min(k,n)}(y)dP(y) \right| \leq \\
&\leq \sup_{k\leq r\leq n} \sup_{k\in\mathbb{N}} k^{\frac{1}{q_0}} \sup_{P(A)\geq\frac{1}{k}} \frac{1}{P(A)} \left| \int_A TX_r(y)dP(y) \right| \lesssim \\
&\lesssim \sup_{1\leq r\leq n} \|TX_r\|_{L_{q_0,\infty}} \leq \sup_{1\leq r\leq n} M_0 \|X_r\|_{L_{p_0,1}} \lesssim M_0 \|X_n\|_{L_{p_0,1}}.
\end{aligned}$$

According to (16) we have

$$\|TX^n\|_{N_{q_0,\infty}(\Phi)} \lesssim M_0 \|X\|_{N_{p_1,1}(F)}$$

and

$$\|T(X-X^n)\|_{N_{q_1,\infty}} = \sup_{k\geq n} k^{\frac{1}{q_1}} \sup_{A\in\Phi_k} \frac{1}{P(A)} \left| \int_{P(A)} T\left(X_k - X_n\right)(y)dP(y) \right| \leq$$

$$\leq \sup_{r\geq n} \sup_{k\in\mathbb{N}} k^{\frac{1}{q_1}} \sup_{P(A)\geq\frac{1}{k}} \frac{1}{P(A)} \left| \int_{P(A)} T\left(X_r - X_n\right) dP \right| \lesssim$$

$$\lesssim \sup_{r\geq n} \|T(X_r - X_n)\|_{L_{q_1,\infty}} \leq M_1 \sup_{r\geq n} \|X_r - X_n\|_{L_{p,q}} =$$

$$= M_1 \|f - X_n\|_{L_{p,q}}.$$

Thus, taking into account (17), we get

$$\|T(X-X^n)\|_{N_{q_1,\infty}(\Phi)} \lesssim M_1 \|X - X_n\|_{N_{p_1,1}(F)},$$

then, by the Theorem 1, we have

$$\|TX\|_{N_{q,s}(\Phi)} \lesssim M_0^{1-\theta} M_1^{\theta} \|X\|_{N_{p,s}(F)},$$

which is equivalent to

$$\|Tf\|_{L_{q,s}} \lesssim M_0^{1-\theta} M_1^{\theta} \|f\|_{L_{p,s}}.$$

In the case when f changes sign, we consider the representation $f = f_+ - f_-$, where $f_+ = \max(f(x),0) \geq 0, f_- = f_+ - f \geq 0$. Using the quasilinearity of the operator T we obtain the statement of the corollary 1.

Corollary 2. *Let $T = \{T_n\}_{n=1}^{\infty}$ be a quasilinear transform such that for any $p \in (a,b)$ and for any $X \in N_{p,1}(F) \cap W(F)$ the following weak inequality holds*

$$\|T(X)\|_{N_{p,\infty}(R)} \leq C_p \|X\|_{N_{p,1}(F)}.$$

Then

$$\|T(X)\|_{N_{p,s}(R)} \leq C_{p,s} \|X\|_{N_{p,s}(F)}, \quad X \in N_{p,s}(F) \cap W(F)$$

for any p and s such that $p \in (a,b)$, $1 \leq s \leq \infty$.

4. Boundedness of some operators in class $N_{p,q}(F)$.

Let $Y = (Y_n, \mathfrak{G}_n)_{n\geq 0}$ be a stochastic sequence and $V = (V_n, \mathfrak{G}_{n-1})$ be a predicted sequence ($\mathfrak{G}_{-1} = \mathfrak{G}_0$). A stochastic sequence $V \cdot Y = ((V \cdot Y)_n, \mathfrak{G}_n)$ such that

$$(V \cdot Y)_n = V_0 Y_0 + \sum_{i=1}^{n} V_i \Delta Y_i,$$

where $\Delta Y_i = Y_i - Y_{i-1}$, is called the transform of Y with respect to V. If Y is a martingale then we say that $V \cdot Y$ is the martingale transform.

Theorem 2. *Let* $0 < q < p < \infty, 1 \leq \tau \leq \infty$ *and* $1/r = 1/q - 1/p$. *Let* $V \cdot Y$ *be a martingale transform of a martingale* Y *by predicted sequence* $V = (V_n, \mathfrak{F}_{n-1})$. *If*

$$\|V\|_{N_{r,\infty}(G)} + \|(n\Delta V_n)\|_{N_{r,\infty}(G)} \leq B,$$

then

$$\|V \cdot Y\|_{N_{q,\tau}(G)} \leq cB \|Y\|_{N_{p,\tau}(G)},$$

where a constant c *depends only on parametres* p, q, τ.

Proof. Let $V \cdot Y$ be a martingale transform of a martingale Y by predicted sequence $V = (V_n, \mathfrak{F}_{n-1})$, i.e.

$$(V \cdot Y)_n = \sum_{k=1}^{n} V_k \Delta Y_k,$$

where $Y_0 = 0, \quad \Delta Y_k = Y_k - Y_{k-1}$. By Abel's transform $(V \cdot Y)_n = \sum_{k=1}^{n-1} \Delta V_k Y_k + V_n Y_n$, we get

$$\|V \cdot Y\|_{N_{q,\infty}(G)} = \sup_{n \in N} n^{-\frac{1}{q}} \overline{(V \cdot Y)_n} \leq \sup_{n \in N} n^{-\frac{1}{q}} \left(\sum_{k=1}^{n} \overline{\Delta V_k Y_k} + \overline{V_n Y_n} \right).$$

Taking into account that $\Delta V_k, Y_k$ are measurable functions with respect to the algebra $\mathfrak{G}_k$, we have $\overline{\Delta V_k Y_k} \leq \overline{\Delta V_k} \ \overline{Y_k}$, $\overline{V_n Y_n} \leq \overline{V_n Y_n}$ and

$$\sup_{n \in N} n^{-\frac{1}{q}} \left(\overline{V \cdot Y}\right)_n \leq \sup_{n \in N} \sum_{k=1}^{n} \left(k^{-\frac{1}{q}} k \overline{\Delta V_k} \right) \left(k^{-\frac{1}{p}-1} \overline{Y_k} \right) + n^{-\frac{1}{r}} \overline{V_n} n^{-\frac{1}{p}} \overline{Y_n} \leq$$

$$\leq \sup_k k^{1-\frac{1}{r}} \overline{\Delta V_k} \cdot \|Y\|_{N_{p,1}(G)} + \|V_n\|_{N_{r,\infty}(G)} \|Y\|_{N_{p,\infty}(G)} \leq$$

$$\leq B \|Y\|_{N_{p,1}(G)}.$$

Hence the weak inequality is proved:

$$\|V \cdot Y\|_{N_{q,\infty}(G)} \leq B \|Y\|_{N_{p,1}(G)}$$

for $1 < q < p < \infty$.

Let $0 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, $0 < q_0 < q < q_1 < \infty$ and $0 < p_0 < p < p_1 < \infty$. Let a pair of numbers (p_0, p_1) and (q_0, q_1) satisfy the following condition

$$\frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q_1} - \frac{1}{p_1} = \frac{1}{q} - \frac{1}{p}. \tag{18}$$

Then from the proved above it follows that

$$\|V \cdot Y\|_{N_{q_0,\infty}(G)} \leq B \|Y\|_{N_{p_0,1}(G)}, \quad \|V \cdot Y\|_{N_{q_1,\infty}(G)} \leq B \|Y\|_{N_{p_1,1}(G)}$$

for $0 < p < q < \infty$.

Taking into account that for any stopping time $k \in N$ processes Y^k and $Y - Y^k$ are martingales, it is possible to apply Theorem 1. Then

$$\|V \cdot Y\|_{N_{q_\theta,\tau}} \le cB \|Y\|_{N_{p_\theta,\tau}},$$

where $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Note that there exists $\theta \in (0,1)$ such that $\frac{1}{p_\theta} = \frac{1}{p}$. Then it follows from (18) that $\frac{1}{q_\theta} = \frac{1}{q}$. The theorem is proved.

Theorem 3. *Let* $0 < p < \infty$,*1*$\le q \le \infty$, $X = (X_n, \mathfrak{G}_n)_{n\ge1} \in W(G)$ *and* $\tau(\omega)$ *be the Markov time,* $X^\tau = (X_{n\wedge\tau}, \mathfrak{G}_n)_{n\ge1}$ *be a stopped process. Then*

$$\|X^\tau\|_{N_{p,q}(G)} \le c\|X\|_{N_{p,q}(G)}.$$

Proof. Denote

$$W_r = \{\omega : \tau(\omega) = r\} \in \mathfrak{G}_r, \quad r = \overline{1, n-1}, \qquad W_n = \{\omega : \tau(\omega) \ge n\} \in \mathfrak{G}_n.$$

Let us show that $\|X^\tau\|_{N_{p,\infty}(G)} \le c\|X\|_{N_{p,1}(G)}$.

If $2^s \le n < 2^{s+1}$, then

$$n^{-\frac{1}{p}} \sup_{A\in\mathfrak{G}_n, P(A)>0} \frac{1}{P(A)} \left| \int_A X^\tau P(d\omega) \right| =$$

$$= n^{-\frac{1}{p}} \sup_{A\in\mathfrak{G}_n, P(A)>0} \frac{1}{P(A)} \left| \sum_{r=1}^{n} \int_{A\cap W_r} X_r P(d\omega) \right| \le$$

$$\le 2^{-\frac{s}{p}} \sup_{A\in\mathfrak{G}_{2^{s+1}-1}} \frac{1}{P(A)} \sum_{t=0}^{s} \sum_{r=2^t}^{2^{t+1}-1} \left| \int_{A\cap W_r} X_r P(d\omega) \right| \le$$

$$\le 2^{-\frac{s}{p}} \sup_{A\in\mathfrak{G}_{2^{s+1}-1}} \frac{1}{P(A)} \sum_{t=0}^{s} \sum_{r=2^t}^{2^{t+1}-1} P(A \cap W_r)\overline{X_r} \le$$

$$\le 2^{-\frac{s}{p}} \sup_{A\in\mathfrak{G}_{2^{s+1}-1}} \frac{1}{P(A)} \sum_{t=0}^{s} \overline{X}_{2^{t+1}-1} \sum_{r=2^t}^{2^{t+1}-1} P(A \cap W_r) =$$

$$= 2^{-\frac{s}{p}} \sup_{A\in\mathfrak{G}_{2^{s+1}-1}} \sum_{t=0}^{s} \overline{X}_{2^{t+1}-1} \sum_{r=2^t}^{2^{t+1}-1} P(W_r|A) \le$$

$$\le \sum_{t=0}^{s} 2^{-\frac{t}{p}} \overline{X}_{2^{t+1}-1} \le 2^{\frac{1}{p}} \sum_{t=0}^{s} 2^{-\frac{t+1}{p}} \overline{X}_{2^{t+1}} =$$

$$= 2^{\frac{1}{p}} \sum_{k=0}^{s+1} 2^{-\frac{k}{p}} \overline{X}_{2^k} \le c\|X\|_{N_{p,1}(G)}.$$

Now, using the Corollary 1 we get the statement of the Theorem 3.

Corollary 3. *Let* $0 < p < \infty$, $1 \le q \le \infty$ *and a process* $X = (X_n, \mathfrak{G}_n)_{n\ge1}$ *be a nonnegative submartingale. Then the process* $X^* = (X_n^*, \mathfrak{G}_n)_{n\ge1}$ *is also submartingale and*

$$\|X^*\|_{N_{p,q}(G)} \sim \|X\|_{N_{p,q}(G)}.$$

Proof. It follows from the Theorem 3 that

$$\|X^*\|_{N_{p,q}(G)} \le c\|X\|_{N_{p,q}(G)}.$$

The reverse inequality is trivial.

5. Interpolation method for stochastic processes.

The interpolation Theorem 1 slightly differs from the classical Marcinkiewicz and Riesz-Thorin type theorems. Therefore for the construction of the interpolation theory of stochastic processes besides the real interpolation method we need some modification of it.

Let $\mathbf{A} = (A_0(F), A_1(F))$ be a pair of quasinormed own subspaces of linear Hausdorff stochastic processes spaces $\mathfrak{N}(F)$, which is defined on a probability space $(\Omega, \mathfrak{F}, P))$ with a filtration $F = \{\mathfrak{F}_n\}_{n\ge1}$. Obviously, this pair is compatible pair and, hence the scale of interpolation spaces is defined with respect to the real method ([1]).

Moreover, let for $0 < \theta < 1$, $0 < q < \infty$

$$(A_0, A_1)_{\theta,q} = \left\{ X \in \mathfrak{N}(F) : \|X\|^q_{(A_0,A_1)_{\theta,q}} = \int_0^\infty (t^{-\theta} K(t,X))^q \frac{dt}{t} < \infty \right\}$$

and for $q = \infty$

$$(A_0, A_1)_{\theta,\infty} = \left\{ X \in \mathfrak{N}(F) : \|X\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,X) < \infty \right\},$$

where

$$K(t, X; A_0, A_1) = \inf_{X = X_0 + X_1} (\|X_0\|_{A_0} + t\|X_1\|_{A_1})$$

is the Peetre functional.

Let $R = \{\tau_k(\omega)\}_{k=1}^\infty$ be a sequence of stopping times with respect to a filtration F, $A(F) = (A_0(F), A_1(F))$ be a pair of quasinormed own subspaces $\mathfrak{N}(F)$. Let us define for $X \in \mathfrak{N}(F)$ and $t \in (0, \infty)$ the following functional

$$K_R(t, X) = K(t, X; A_0, A_1, R) = \inf_{\tau \in R} (\|X - X^\tau\|_{A_0} + t\|X^\tau\|_{A_1}).$$

Here the infimum is taken over all stopping times from R. Moreover for $0 < q < \infty$

$$(A_0, A_1)^R_{\theta,q} = \left\{ X \in \mathfrak{N}(F) : \|X\|^q_{(A_0,A_1)_{\theta,q}} = \int\limits_0^\infty (t^{-\theta} K_R(t,X))^q \frac{dt}{t} < \infty \right\},$$

and for $q = \infty$

$$(A_0, A_1)^R_{\theta,\infty} = \left\{ X \in \mathfrak{N}(F) : \|X\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K_R(t,X) < \infty \right\}.$$

Theorem 4. *Let $(A_0(F), A_1(F)), (B_0(\Phi), B_1(\Phi))$ be two compatible pairs of stochastic processes and $R = \{\tau(\omega)\}$ be some fixed family of Markov times with respect to a filtration F. If T is a quasilinear map for stochastic processes $X = (X_n, \mathfrak{F}_n)_{n\geq 1}$ and*

$$\|T(X - X^\tau)\|_{B_0} \leq M_0 \|X - X^\tau\|_{A_0}, \quad \|T(X^\tau)\|_{B_1} \leq M_1 \|X^\tau\|_{A_1},$$

for all stopping times $\tau \in R$, then

$$\|T(X)\|_{\mathbf{B}_{\theta,\mathbf{q}}} \leq C M_0^{1-\theta} M_1^\theta \|X\|_{\mathbf{A}^R_{\theta,q}},$$

where the constant C is from the definition of quasilinearity of the operator T.

Proof.

$$\|T(X)\|_{\mathbf{B}_{\theta,\mathbf{q}}} = \left(\int_0^\infty \left(t^{-\theta} \inf_{T(X)=Y_0+Y_1} (\|Y_0\|_{B_0} + t\|Y_1\|_{B_1}) \right)^q \frac{dt}{t} \right)^{1/q} \leq$$

$$\leq \left(\int_0^\infty \left(t^{-\theta} \inf_{\tau\in R} (\|T(X) - T(X^\tau)\|_{B_0} + t\|T(X^\tau\|_{B_1}) \right)^q \frac{dt}{t} \right)^{1/q} \leq$$

$$\leq C \left(\int_0^\infty \left(t^{-\theta} \inf_{\tau\in R} (\|T(X - X^\tau)\|_{B_0} + t\|T(X^\tau\|_{B_1}) \right)^q \frac{dt}{t} \right)^{1/q} \leq$$

$$\leq C M_0 \left(\int_0^\infty \left(t^{-\theta} \inf_{\tau\in R} (\|X - X^\tau\|_{A_0} + t\frac{M_1}{M_0}\|X^\tau\|_{A_1}) \right)^q \frac{dt}{t} \right)^{1/q} =$$

$$= C M_0^{1-\theta} M_1^\theta \|X\|_{\mathbf{A}^R_{\theta,q}(F)}.$$

The theorem is proved.

Lemma 5. *Let $a > 1$ and $R = \{k\}_{k\in\mathbb{N}}$ be stopping times. Then for $0 < q < \infty$*

$$\|X\|_{(A_0,A_1)^R_{\theta,q}} \sim \left(\sum_{n=0}^\infty (a^{\theta n} K_R(a^n, X))^q \right)^{1/q}$$

and

$$\|X\|_{(A_0,A_1)^R_{\theta,\infty}} \sim \sup_n a^{\theta n} K_R(a^n, X).$$

The proof is similar to the proof of the Lemma 3.

Theorem 5. *Let* $1 < p_0 < p_1 < \infty$, $1 \le q_0, q_1, q \le \infty$, $0 < \theta < 1$, $\frac{1}{q} = \frac{1\theta}{p_0} + \frac{\theta}{p_1}$ *and* $R = \{k\}_{k\in\mathbb{N}}$ *be the stopping times. Then for any stochastic process* $X = (X_n, \mathfrak{F}_n)_{n\ge1}$,

$$\|X\|_{N_{p,q}(F)} \le c\|X\|_{(N_{p_0,q_0}(F),N_{p_1,q_1}(F))_{\theta,q}},$$

where the constant c *depends only on parametres* p_i, q_i, θ, $i = 0, 1$.

If $X = (X_n, \mathfrak{F}_n)_{n\ge1} \in W(F)$, *then*

$$\|X\|_{(N_{p_0,q_0}(F),N_{p_1,q_1}(F))^R_{\theta,q}} \le c\|X\|_{N_{p,q}(F)},$$

where the constant c *also depends only on parametres* p_i, q_i, θ, $i = 0, 1$.

Proof. Let $X = Y + Z$ be any representation of a process X, where $Y \in N_{p_0,q_0}(F)$, $Z \in N_{p_1,q_1}(F)$. To prove the first statement of the theorem, we use the following inequality: $\overline{X}_m \le \overline{Y}_m + \overline{Z}_m$.

For any $a > 1$ we have

$$\begin{aligned} a^{-\frac{n}{p}}\overline{X}_{a^n} &\le a^{-\frac{n}{p}+\frac{n}{p_0}}\left(a^{-\frac{n}{p_0}}\overline{Y}_{a^n} + a^{-\frac{n}{p_0}+\frac{n}{p_1}}a^{-\frac{n}{p_1}}\overline{Z}_{a^n}\right) \le \\ &\le a^{-\frac{n}{p}+\frac{n}{p_0}}\left(\sup_n a^{-\frac{n}{p_0}}\overline{Y}_{a^n} + a^{-\frac{n}{p_0}+\frac{n}{p_1}}\sup_n a^{-\frac{n}{p_1}}\overline{Z}_{a^n}\right) = \\ &= a^{-\frac{n}{p}+\frac{n}{p_0}}\left(\| Y \|_{N_{p_0,\infty}(F)} + a^{-\frac{n}{p_0}+\frac{n}{p_1}} \| Z \|_{N_{p_1,\infty}(F)}\right). \end{aligned}$$

By putting $a = 2^{\frac{p_0p_1}{p_1-p_0}}$, we get $a^{-\frac{n}{p}}\overline{X}_n \le 2^{n\theta}K(2^n, X)$. Therefore, using (3) and Lemma 6, we have

$$\begin{aligned} \|X\|_{N_{p,q}(F)} &\sim \left(\sum_{n=1}^{\infty}\left(a^{-\frac{n}{p}}\overline{X}_n\right)^q\right)^{\frac{1}{q}} \le \\ &\le \left(\sum_{n=1}^{\infty}\left(a^{n\theta}K(a^n, X)\right)^q\right)^{\frac{1}{q}} \sim \|X\|_{(N_{p_0,\infty}(F),N_{p_1,\infty}(F))_{\theta q}}. \end{aligned}$$

Let us prove the second statement of the theorem. Let $X = (X_n, \mathfrak{F}_n)_{n\ge1} \in W(F)$.

By using Lemma 6 and Lemma 2 we have

$$\|X\|_{(N_{p_0,q_0}(F),N_{p_1,q_1}(F))^R_{\theta q}} \sim \left(\sum_{n=0}^{\infty}(2^{\theta n}K(2^n,X))^q\right)^{1/q} =$$

$$= \left(\sum_{n=0}^{\infty}(2^{\theta n}\inf_k(\|X-X^k\|_{N_{p_0,q_0}}+2^{-n}\|X^k\|_{N_{p_1,q_1}}))^q\right)^{1/q} \leq$$

$$\leq c\left(\sum_{n=0}^{\infty}(2^{\theta n}\inf_k(\|X-X^k\|_{N_{p_0,1}}+2^{-n}\|X^k\|_{N_{p_1,1}}))^q\right)^{1/q}.$$

Let $k=2^{n\gamma}$, then

(19)

$$\|X\|_{(N_{p_0,q_0}(F),N_{p_1,q_1}(F))^R_{\theta q}} \leq c\left(\sum_{n=0}^{\infty}(2^{\theta n}(\|X-X^{2^{n\gamma}}\|_{N_{p_0,1}}+2^{-n}\|X^{2^{n\gamma}}\|_{N_{p_1,1}}))^q\right)^{1/q}.$$

Further, we have

$$\|X-X^{2^{n\gamma}}\|_{N_{p_0,1}} = \sum_{k=1}^{\infty}k^{-1\frac{1}{p_0}}\overline{(X-X^{2^{n\gamma}})}_k \sim \sum_{k=0}^{\infty}2^{-\frac{k}{p_0}}\overline{(X-X^{2^{n\gamma}})}_{2^k} =$$

$$(20) \qquad == \sum_{k=n\gamma}^{\infty}2^{-\frac{k}{p_0}}\overline{(X-X^{2^{n\gamma}})}_{2^k} \leq (1+C)\sum_{k=n\gamma}^{\infty}2^{-\frac{k}{p_0}}\overline{X}_{2^k},$$

and

$$\|X^{2^{n\gamma}}\|_{N_{p_1,1}} = \sum_{k=1}^{\infty}k^{-1\frac{1}{p_1}}\overline{X^{2^{n\gamma}}_k} \sim \sum_{k=0}^{\infty}2^{-\frac{k}{p_1}}\overline{X^{2^{n\gamma}}_{2^k}} \leq$$

(21)

$$\leq \sum_{k=0}^{n\gamma}2^{-\frac{k}{p_1}}\overline{X}_{2^k}+\overline{X}_{2^{n\gamma}}\sum_{k=n\gamma}^{\infty}2^{-\frac{k}{p_1}} \leq c\left(\sum_{k=0}^{n\gamma}2^{-\frac{k}{p_1}}\overline{X}_{2^k}+2^{n\gamma(\frac{1}{p_0}-\frac{1}{p_1})}\sum_{k=n\gamma}^{\infty}2^{-\frac{k}{p_0}}\overline{X}_{2^k}\right).$$

By using Lemma 4 for $\frac{1}{\gamma}=\frac{1}{p_0}-\frac{1}{p_1}$, (20), and (21), we have

$$\left(\sum_{n=0}^{\infty}(2^{\theta n}\|X-X^{2^{n\gamma}}\|_{N_{p_0,1}})^q\right)^{1/q} \leq c\left(\sum_{n=0}^{\infty}(2^{\theta n}\sum_{k=n\gamma}^{\infty}2^{-\frac{k}{p_0}}\overline{X}_{2^k})^q\right)^{1/q} \leq$$

$$(22) \qquad \leq c_1\left(\sum_{k=0}^{\infty}(2^{-\frac{k}{p}}\overline{X}_{2^k})^q\right)^{1/q} \sim \|X\|_{N_{p,q(F)}}$$

and

$$\left(\sum_{n=0}^{\infty} 2^{-(1-\theta)nq} \|X^{2^{n\gamma}}\|^q_{N_{p_1,1}}\right)^{1/q} \le$$

$$\le c\left(\sum_{n=0}^{\infty} 2^{-nq(1-\theta)}\left(\sum_{k=0}^{n\gamma} 2^{-\frac{k}{p_1}}\overline{X}_{2^k} + 2^{n\gamma(\frac{1}{p_0}-\frac{1}{p_1})}\sum_{k=n\gamma}^{\infty} 2^{-\frac{k}{p_0}}\overline{X}_{2^k}\right)^q\right)^{1/q} \le$$

$$\le c_1\left(\sum_{k=0}^{\infty}(2^{-\frac{k}{p}}\overline{X}_{2^k})^q\right)^{1/q} \sim \|X\|_{N_{p,q}}(F). \tag{23}$$

By applying Minkowski's inequality to (19) and using estimates (22) and (23), we obtain

$$\|X\|_{(N_{p_0,q_0},N_{p_1,q_1})^R_{\theta q}} \le c\|X\|_{N_{p,q}(F)}.$$

The theorem is proved.

6. The space $N_p^{\alpha,q}(F)$, the embedding theorems.

Let us remind the definition of $N_p^{\alpha,q}(F)$ space. Let $1 < p < \infty, 0 < q \le \infty$, $\alpha \ge 0$ and $p' = \frac{p}{p-1}$. The stochastic process $X = (X_n, \mathfrak{F}_n)_{n\ge1}$ belongs to space $N_p^{\alpha q}(F)$, if $X = (X_n, \mathfrak{F}_n)_{n\ge1}$ is a martingale and for $0 < q < \infty$

$$\|X\|_{N_p^{\alpha,q}(F)} = \left(\sum_{k=0}^{\infty}(2^{\alpha k}\overline{\Delta X_k})^q\right)^{1/q} < \infty,$$

and for $q = \infty$

$$N_p^{\alpha,\infty}(F) = \{X = (X_n, \mathfrak{F}_n)_{n\ge1} : \sup_k 2^{\alpha k}\overline{\Delta X_k} < \infty\},$$

where

$$\overline{\Delta X}_k = \sup_{A\in\mathfrak{F},P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}}\left|\int_A (X_{2^k} - X_{2^{k-1}})P(d\omega)\right|.$$

We also set that $X_{\frac{1}{2}}(\omega) \equiv 0$.

Lemma 6. *Let $\alpha > 0$, $0 < q \le \infty$, $1 < p \le \infty$. If $X = (X_n, \mathfrak{F}_n)_{n\ge1} \in N_p^{\alpha,q}(F) \cap W(F)$, then there exists a random variable X_∞ such that $X_n \underset{n\to\infty}{\longrightarrow} X_\infty$ (a.p.).*

Proof. Let $L_{p\infty}(\Omega)$ be the Marcinkiewicz–Lorentz space and $2^{\nu-1} \leq n < 2^{\nu}$. Using the equivalent norm of $L_{p,\infty}(\Omega)$ spaces (see [14]) and measurability of function X_n with respect to σ - algebra $\mathfrak{F}_n$, we get

$$\|X_n\|_{L_{p,\infty}[0,1]} \sim \sup_{A\in\mathfrak{F},P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A X_n P(d\omega)\right| =$$

$$= \sup_{A\in\mathfrak{F}_n,P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A X_n P(d\omega)\right| \leq$$

$$\leq C \sup_{A\in\mathfrak{F}_{2^\nu},P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A X_{2^\nu} P(d\omega)\right| \leq$$

$$\leq C \sum_{k=0}^{\nu-1} \sup_{A\in\mathfrak{F}_{2^k},P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A (X_{2^{k+1}} - X_{2^k}) P(d\omega)\right| \leq$$

$$\leq C \|X\|_{N_p^{0,1}(F)}.$$

Taking into account that $N_p^{\alpha,q}(F) \hookrightarrow N_p^{0,1}(F)$, for $\alpha>0$, we have $\|X_n\|_{L_{p,\infty}[0,1]} \leq c\|X_n\|_{N_p^{\alpha,q}(F)}$.

But, $M|X_n| \leq c\|X_n\|_{L_{p,\infty}[0,1]}$, therefore by the Doob theorem ([8]), the process X_n converges almost surely.

Lemma 7. *Let $\alpha > 0$, $0 < q \leq \infty$, $1 < p \leq \infty$ and $X = (X_n, \mathfrak{F}_n)_{n\geq 1} \in W(F)$. Then*

$$\|X\|_{N_p^{\alpha,q}(F)} \sim \left(\sum_{k=0}^{\infty} (2^{\alpha k}\widetilde{X_k})^q\right)^{1/q},$$

where

$$\widetilde{X_k} = \sup_{A\in\mathfrak{F},P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A (X_\infty - X_{2^{k-1}}) P(d\omega)\right|.$$

Proof. An existence of X_∞ follows from Lemma 6. Further, we have

$$\widetilde{X_k} = \sup_{A\in\mathfrak{F}_{2^k}} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A \sum_{m=k}^{\infty} (X_{2^m} - X_{2^{m-1}}) P(d\omega)\right| \leq$$

$$\leq \sum_{m=k}^{\infty} \sup_{A\in\mathfrak{F}_{2^m}} \frac{1}{(P(A))^{\frac{1}{p'}}} \left|\int_A (X_{2^m} - X_{2^{m-1}}) P(d\omega)\right| = \sum_{m=k}^{\infty} \overline{\Delta X}_m.$$

Therefore, using Lemma 4, we obtain

$$\left(\sum_{k=0}^{\infty}(2^{\alpha k}\widetilde{X_k})^q\right)^{1/q} \leq \left(\sum_{k=0}^{\infty}(2^{\alpha k}\sum_{m=k}^{\infty}\overline{\Delta X}_m)^q\right)^{1/q} \leq$$

$$\leq \left(\sum_{m=0}^{\infty}2^{m\alpha q}\overline{\Delta X}_m^q\right)^{1/q} \sim \|X\|_{N_p^{\alpha,q}(F)}.$$

The reverse inequality follows from the expression:

$$\overline{\Delta X}_k = \sup_{P(A)>0}\frac{1}{(P(A))^{\frac{1}{p'}}}\left|\int_A (X_{2^k}-X_{2^{k-1}})P(d\omega)\right| \leq$$

$$\leq \sup_{P(A)>0}\frac{1}{(P(A))^{\frac{1}{p'}}}\left|\int_A (X_{2^k}-X_{\infty})P(d\omega)\right| +$$

$$+ \sup_{A\in\mathfrak{F}_{2^k},P(A)>0}\frac{1}{(P(A))^{\frac{1}{p'}}}\left|\int_A (X_{\infty}-X_{2^{k-1}})P(d\omega)\right|.$$

The lemma is proved.

Theorem 6. *Let* $1 < p, q, q_0, q_1 \leq \infty$, $\alpha_0 < \alpha_1$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $R = \{r\}_{r\in\mathbb{N}}$. *Then*

$$(N_p^{\alpha_0,q_0}(F), N_p^{\alpha_1,q_1}(F))_{\theta q}^R = N_p^{\alpha,q}(F).$$

Proof. Using Lemma 5 and Lemma 7 we have

$$\|X\|_{(N_p^{\alpha_0,q_0},N_p^{\alpha_1,q_1})_{\theta,q}^R} \sim \left(\sum_{n=0}^{\infty}(2^{\theta n}K(2^n,X))^q\right)^{1/q} =$$

$$= \left(\sum_{n=0}^{\infty}(2^{\theta n}\inf_r(\|X-X^{2^r}\|_{N_p^{\alpha_0,q_0}} + 2^{-n}\|X^{2^r}\|_{N_p^{\alpha_1,q_1}}))^q\right)^{1/q} \leq$$

$$\leq c\left(\sum_{n=0}^{\infty}(2^{\theta n}\inf_r(\|X-X^{2^r}\|_{N_p^{\alpha_0,1}} + 2^{-n}\|X^{2^r}\|_{N_p^{\alpha_1,1}}))^q\right)^{1/q}.$$

Putting $r = n\gamma$, $\gamma = \alpha_1 - \alpha_0$ we get

(24)

$$\|X\|_{(N_p^{\alpha_0,q_0},N_p^{\alpha_1,q_1})_{\theta,q}^R} \leq c\left(\sum_{n=0}^{\infty}(2^{\theta n}(\|X-X^{2^{n\gamma}}\|_{N_p^{\alpha_0,1}} + 2^{-n}\|X^{2^{n\gamma}}\|_{N_p^{\alpha_1,1}}))^q\right)^{1/q}.$$

It follows from the definition that $\overline{(\Delta X^{2^r})}_k = \overline{\Delta X}_k$ for $k = 0, 1, \ldots, r-1$; $\overline{(\Delta X^{2^r})}_k = 0$ for $k \geq r$, $\overline{\Delta(X - X^{2^r})}_k = 0$ for $k = 0, 1, \ldots, r-1$; $\overline{\Delta(X - X^{2^r})}_k =$

$\overline{\Delta X}_k$ for $k \geq r$, therefore,

$$\|X^{2^r}\|_{N_p^{\alpha,1}(F)} = \sum_{k=0}^{r-1} 2^{\alpha k}\overline{\Delta X}_k,$$

$$\|X - X^{2^r}\|_{N_p^{\alpha,1}(F)} = \sum_{k=r}^{\infty} 2^{\alpha k}\overline{\Delta X}_k.$$

Substituting these equalities in (24) and applying Lemma 4, we get

$$\|X\|_{(N_p^{\alpha_0 q_0}, N_p^{\alpha_1 q_1})^R_{\theta q}} \leq c \left(\sum_{n=0}^{\infty} 2^{\theta n q} \left(\sum_{k=n\gamma}^{\infty} 2^{\alpha_0 k}\overline{\Delta X}_k + \sum_{k=0}^{n\gamma-1} 2^{\alpha_1 k}\overline{\Delta X}_k \right)^q \right)^{1/q} \leq$$

$$\leq c \left[\left(\sum_{n=0}^{\infty} \left(2^{(\alpha_0+\theta\gamma)n}\overline{\Delta X}_k\right)^q \right)^{1/q} + \left(\sum_{n=0}^{\infty} \left(2^{(\alpha_1+(1-\theta)\gamma)n}\overline{\Delta X}_k\right)^q \right)^{1/q} \right] = 2c\|X\|_{N_p^{\alpha,q}}.$$

For the proof of reverse estimate we use the fact that for any r and k the following equality holds

$$\overline{\Delta X}_k = \overline{\Delta (X - X^{2^r})}_k + \overline{(\Delta X^{2^r})}_k.$$

Then we have

$$\|X\|_{N_p^{\alpha,q}} = \left(\sum_{k=0}^{\infty} (2^{\alpha k}\overline{\Delta X}_k)^q \right)^{1/q} \leq$$

$$\leq \left(\sum_{k=0}^{\infty} \left(2^{\alpha k - \alpha_0 k} (\sup_k 2^{\alpha_0 k}\overline{\Delta (X - X^{2^r})}_k + 2^{\alpha_0 k - \alpha_1 k} \cdot \sup_k 2^{\alpha_1 k}\overline{(\Delta X^{2^r})}_k) \right)^q \right)^{1/q} =$$

$$= \left(\sum_{k=0}^{\infty} \left(2^{\alpha k - \alpha_0 k} (\|X - X^{2^r}\|_{N_p^{\alpha_0,\infty}} + 2^{\alpha_0 k - \alpha_1 k}\|X^{2^r}\|_{N_p^{\alpha_1,\infty}}) \right)^q \right)^{1/q}.$$

Substituting $2^{\alpha_1 - \alpha_0} = a$ and using Lemma 3, we have

$$\|X\|_{N_p^{\alpha,q}} \leq \left(\sum_{k=0}^{\infty} (a^{\theta k} \cdot K(a^k, X, N_p^{\alpha_0,\infty}, N_p^{\alpha_1,\infty}))^q \right)^{1/q} \sim \|X\|_{(N_p^{\alpha_0,\infty}, N_p^{\alpha_1,\infty})^R_{\theta q}},$$

since $(N_p^{\alpha_0,q_o}, N_p^{\alpha_1,q_1})^R_{\theta q} \hookrightarrow (N_p^{\alpha_0,\infty}, N_p^{\alpha_1,\infty})^R_{\theta q}$, the proof is complete.

Theorem 7. *Let* $1 < r \leq p < \infty$,*1*$\leq q \leq \infty$, $\alpha = \frac{1}{r} - \frac{1}{p}$ *and the filtration* $F = \{\mathfrak{F}_n\}_{n\geq 1}$ *be such that for every* $k = 1, 2...$ *and for all* $A \in \mathfrak{F}_k$ *the following condition holds*

$$P(A) \geq \frac{C}{k}, \tag{25}$$

where the constant $C > 0$ does not depend on k. Then

$$N_r^{\alpha,q}(F) \hookrightarrow N_{p,q}(F).$$

Proof. Let us show that

$$N_r^{\alpha,1}(F) \hookrightarrow N_{p,\infty}(F). \tag{26}$$

$$\|X\|_{N_{p,\infty}(F)} \sim \sup_k 2^{-\frac{k}{p}}\overline{X}_{2^k} = \sup_k 2^{-\frac{k}{p}} \sup_{A\in\mathfrak{F}_{2^k}} \frac{1}{P(A)}\left|\int_A X_{2^k}P(d\omega)\right| \sim$$

$$\sim \sup_k 2^{-\frac{k}{p}} \sup_{A\in\mathfrak{F}_{2^k}} \frac{1}{P(A)}\left|\sum_{m=0}^{k-1}\int_A (X_{2^m} - X_{2^{m-1}})P(d\omega)\right| \leq$$

$$\leq \sup_k \sum_{m=0}^{k-1} 2^{-\frac{m}{p}} \sup_{A\in\mathfrak{F}_{2^m}} \frac{1}{(P(A))^{\frac{1}{r'}}}\frac{1}{(P(A))^{\frac{1}{r}}}\left|\int_A (X_{2^m} - X_{2^{m-1}})P(d\omega)\right|.$$

According to the condition (25), for $A \in \mathfrak{F}_{2^m}$ we have that $P(A) > \frac{C}{2^m}$. Therefore for $\alpha = \frac{1}{r} - \frac{1}{p}$ we get

$$\|X\|_{N_{p,\infty}(F)} \leq C\sup_k \sum_{m=0}^{k-1} 2^{-\frac{m}{p}+\frac{m}{r}} \sup_{A\in\mathfrak{F}_{2^m}} \frac{1}{(P(A))^{\frac{1}{r'}}}\left|\int_A (X_{2^m} - X_{2^{m-1}})P(d\omega)\right| \sim$$

$$\sim \|X\|_{N_r^{\alpha,1}}.$$

Thus, (26) is proved.

Now, let $\alpha_0 < \alpha_1$, $1 < p_0 < p_1 < \infty$ and $\theta \in (0,1)$ such that

$$\frac{1}{p_0} + \alpha_0 = \frac{1}{p_1} + \alpha_1 = \frac{1}{r}, \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then using interpolation Theorems 6 and 5 we obtain

$$(N_r^{\alpha_0,1}(F), N_r^{\alpha_1,1}(F))_{\theta,q} = N_r^{\alpha,q}(F),$$

$$(N_{p_0,\infty}(F), N_{p_1,\infty}(F))_{\theta,q} = N_{p,q}(F).$$

It follows from (29) that $(N_r^{\alpha_0,1}(F), N_r^{\alpha_1,1}(F))_{\theta,q} \hookrightarrow (N_{p_0,\infty}(F), N_{p_1,\infty}(F))_{\theta q}$. Hence, $N_r^{\alpha,q}(F) \hookrightarrow N_{p,q}(F)$, where $\alpha = \frac{1}{r} - \frac{1}{p}$. The proof is complete.

7. Spaces with variable approximation properties by Haar system.

In this paragraph we consider some applications of the introduced interpolation method to Besov type spaces with variable approximation properties.

Let $\Omega = [0,1]$, $\mathfrak{F}$ be a $\sigma-$ algebra of Borel subsets of set Ω, P be a linear Lebesgue measure on $\mathfrak{F}$, $F = \{\mathfrak{F}_n\}_{n\geq 1}$ be the Haar filtration, $R = \{\tau_k\}_{k=0}^{\infty}$ be a sequence of stopping times such that for any $k \geq 0$ the following conditions hold: $\tau_k \leq \tau_{k+1}$ (a.p.) and

$$\lim_{k\to\infty} \tau_k(\omega) = \infty \quad \text{(a.p.)} \tag{30}$$

For a function $f(x) \in L[0,1]$ we denote by $\{c_k(f)\}_{k\geq 1}$ the Fourier coefficients by Haar functions system $\{H_k(x)\}_{k\geq 1}$ ([9]). For the given stopping time $\tau_k(\omega)$ we denote

$$S(f,\tau_k)(\omega) = \sum_{m=1}^{\tau_k(\omega)} c_m(f) H_m(\omega),$$

which we call the Fourier-Haar partial sum of a function f, corresponding to the Markov time τ_k.

Let $1 < p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$. By $B_p^{\alpha q}[R]$ we denote the set of functions $f \in L[0,1]$, for which

$$\|f\|_{B_p^{\alpha q}[R]} = \left(\sum_{k=0}^{\infty} 2^{\alpha k q} \left\| f - S(f,\tau_k) \right\|_{L_p} \right)^{\frac{1}{q}} < \infty$$

for $0 < q < \infty$,

$$\|f\|_{B_p^{\alpha\infty}[R]} = \sup_k 2^{\alpha k} \left\| f - S(f,\tau_k) \right\|_{L_p} < \infty,$$

for $q = \infty$.

Conceptually, the introduced spaces are close to spaces with variable smoothness. Here we mention works of H.-G.Leopold ([10]), F.Cobos, D.L.Fernandez ([7]) and O.V.Besov ([2]-[5]).

Lemma 8. *Let $1 < p < \infty$, $f \in L_p[0,1]$, $S(f,\tau)$ be the Fourier - Haar partial sum with respect to the Markov time τ. Then*

$$\|S(f,\tau)\|_p \leq c \|f\|_p .$$

Proof. Denote

$$\mathfrak{F}_\tau = \{A \in \mathfrak{F} : \quad A \cap \{\tau = n\} \in \mathfrak{F}_n \quad \text{for every} \quad n \geq 1\}.$$

Let $L_{p,\infty}[0,1]$ be the Marcinkiewicz–Lorentz space. Using the equivalent norm of $L_{p,\infty}[0,1]$ spaces (see [14]) and martingale properties of Fourier - Haar partial sums we get

$$\|S(f,\tau)\|_{L_{p,\infty}[0,1]} \sim \sup_{A\in\mathfrak{F}, P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left| \int_A S(f,\tau) P(d\omega) \right| =$$

$$= \sup_{A\in\mathfrak{F}_\tau, P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left| \int_A S(f,\tau) P(d\omega) \right| =$$

$$= \sup_{A\in\mathfrak{F}_\tau, P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left| \int_A f(\omega) P(d\omega) \right| \leq$$

$$\leq \sup_{A\in\mathfrak{F}, P(A)>0} \frac{1}{(P(A))^{\frac{1}{p'}}} \left| \int_A f(\omega) P(d\omega) \right| = \|f\|_{L_{p,\infty}[0,1]} .$$

Now, applying the interpolation theorem (see [1]), we obtain the statement of the lemma.

Lemma 9. *Let* $1 < p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ *and*

$$\Delta(f, \tau_k) = \sum_{r=\tau_k}^{\tau_{k+1}-1} c_r(f) H_r(\omega).$$

Then

$$\|f\|_{B_p^{\alpha,q}[R]} \sim \left(\sum_{k=0}^{\infty} 2^{\alpha k q} \|\Delta(f, \tau_k)\|_{L_p}^q \right)^{1/q}.$$

Proof. Proof is similar to the proof of Lemma 7.

Theorem 8. *Let* $1 \leq q_0, q_1, q \leq \infty, 0 < \alpha_0 < \alpha_1, 0 < \theta < 1, \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. *Then*

$$(B_p^{\alpha_0,q_0}[R], B_p^{\alpha_1,q_1}[R])_{\theta,q}^R = B_p^{\alpha,q}[R].$$

Proof. By using Lemma 9 we have

$$\|f\|_{(B_p^{\alpha_0,q_0}[R], B_p^{\alpha_1,q_1}[R])_{\theta,q}^R} \leq c\|f\|_{(B_p^{\alpha_0,1}[R], B_p^{\alpha_1,1}[R])_{\theta,q}^R} \sim$$

$$\sim \left(\sum_{n=1}^{\infty} 2^{\theta q n} \inf_r \left(\sum_{k=r}^{\infty} 2^{\alpha_0 k} \|\Delta(f, \tau_k)\|_{L_p} + 2^{-n} \sum_{k=0}^{r-1} 2^{\alpha_1 k} \|\Delta(f, \tau_k)\|_{L_p}) \right)^q \right)^{1/q} \leq$$

$$\leq \left(\sum_{n=1}^{\infty} 2^{\theta q n} \left(\sum_{k=n\gamma}^{\infty} 2^{\alpha_0 k} \|\Delta(f, \tau_k)\|_{L_p} + 2^{-n} \sum_{k=0}^{n\gamma-1} 2^{\alpha_1 k} \|\Delta(f, \tau_k)\|_{L_p} \right)^q \right)^{1/q} \leq$$

$$\leq \left(\sum_{n=1}^{\infty} \left(2^{\theta n} \sum_{k=n\gamma}^{\infty} 2^{\alpha_0 k} \|\Delta(f, \tau_k)\|_{L_p} \right)^q \right)^{1/q} +$$

$$+ \left(\sum_{n=1}^{\infty} \left(2^{-(1-\theta)n} \sum_{k=0}^{n\gamma-1} 2^{\alpha_1 k} \|\Delta(f, \tau_k)\|_{L_p} \right)^q \right)^{1/q}$$

By applying Lemma 4 we get

$$\|f\|_{(B_p^{\alpha_0,q_0}[R], B_p^{\alpha_1,q_1}[R])_{\theta q}^R} \leq c \|f\|_{B_p^{\alpha,q}[R]}.$$

Let us prove the reverse embedding. Let $f \in (B_p^{\alpha_0,q_0}[R], B_p^{\alpha_1,q_1}[R])_{\theta,q}^R$, $f = f_0 + f_1$ be an arbitrary representation of a function, $f_0 \in B_p^{\alpha_0,q_0}[R]$ and $f_1 \in B_p^{\alpha_1,q_1}[R]$.

Then

$$2^{\alpha k}\|\Delta(f,\tau_k)\|_{L_p} \le 2^{\alpha k}(\|\Delta(f_0,\tau_k)\|_{L_p} + \|\Delta(f_1,\tau_k)\|_{L_p}) \le$$
$$\le 2^{(\alpha-\alpha_0)k}(\sup_{r\in\mathbb{N}} 2^{\alpha_0 k}\|\Delta(f_0,\tau_k)\|_{L_p} + 2^{(\alpha_0-\alpha_1)k}\sup_{r\in\mathbb{N}} 2^{\alpha_1 k}\|\Delta(f_1,\tau_k)\|_{L_p}) =$$
$$= 2^{(\alpha-\alpha_0)k}(\|f_0\|_{B_p^{\alpha_0,\infty}[R]} + 2^{(\alpha_0-\alpha_1)k}\|f_1\|_{B_p^{\alpha_1,\infty}[R]}).$$

Since the representation $f = f_0 + f_1$ is arbitrary, we have

$$2^{\alpha k}\|\Delta(f,\tau_k)\|_{L_p} \le 2^{(\alpha-\alpha_0)k}K_R(f, 2^{(\alpha_0-\alpha_1)k}; B_p^{\alpha_0,\infty}[R], B_p^{\alpha_1,\infty}[R]).$$

Hence, putting $a = 2^{\alpha_0-\alpha_1}$ we get

$$\|f\|_{B_p^{\alpha,q}[R]} \le \left(\sum_{k=0}^{\infty}\left(2^{(\alpha_0-\alpha_1)\theta k}K_R(f, 2^{(\alpha_0-\alpha_1)k})\right)^q\right)^{1/q} =$$
$$= \left(\sum_{k=0}^{\infty}\left(a^{\theta k}K_R(f, a^k)\right)^q\right)^{1/q} \sim$$
$$\sim \|f\|_{(B_p^{\alpha_0,\infty}[R], B_p^{\alpha_1,\infty}[R])^R_{\theta,q}} \le c\|f\|_{(B_p^{\alpha_0,q_0}[R], B_p^{\alpha_1,q_1}[R])^R_{\theta,q}}.$$

The theorem is proved.

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