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Interpolation of Besov $B_{p\tau}^{\sigma q}$ and Lizorkin–Triebel $F_{p\tau}^{\sigma q}$ spaces

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Abstract. An interpolation method is introduced for anisotropic spaces which generalizes the method by D. L. Fernandez [4]. By means of this method, interpolation properties of Besov $B_{p\tau}^{\sigma q}$ and Lizorkin–Triebel $F_{p\tau}^{\sigma q}$ spaces are investigated. Among others, the completeness of the scale of these spaces is proved with respect to the considered interpolation method.

Introduction

The first interpolation theorems for Sobolev and Besov spaces have appeared in Peetre's work [9]. In particular, he showed that for $1 \le p \le \infty$, $\alpha_1 \in \mathbb{N}$, $0 < \theta < 1$ and $1 \le q \le \infty$ the equality

$$(L_p, W_p^{\alpha_1})_{\theta q} = B_p^{\alpha q}, \qquad \alpha = \theta\alpha_1$$

is correct.

Let's note Lions and Peetre's work [7], in which the following equality

$$(B_{p_0}^{\alpha_0 q_0}, B_{p_1}^{\alpha_1 q_1})_{\theta q} = B_p^{\alpha q},$$

is studied for

$$1 \le p_0 \ne p_1 \le \infty, \quad 0 < \alpha_0 \ne \alpha_1 < \infty, \quad 1 \le q_0 \ne q_1 \le \infty$$

and

$$0 < \theta < 1, \quad 1 \le q \le \infty, \quad \text{where} \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1,$$

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is correct only in case

$$q = p \quad \text{and} \quad 1/p = (1-\theta)/p_0 + \theta/p_1 = (1-\theta)/q_0 + \theta/q_1$$

(so-called diagonal case). In a nondiagonal case, the result of interpolation of Besov spaces is not described by spaces from same scale.

Further study of interpolation properties of Besov and Lizorkin–Triebel spaces, concerning the classical complex and real methods, the works of many authors were devoted, with review it is possible to familiarize in the monographies of Berg and Levstrem [3], Triebel [10].

In Krepkogorskii's works [5], [6], interpolation properties of Besov and Lizorkin–Triebel spaces in a nondiagonal case and concerning Sparr method were studied as follows. By Asecritova's and others [1]: interpolation properties of Besov and Lizorkin–Triebel spaces concerning Sparr method, by the present authors [2]: interpolation properties of Besov spaces concerning a method of multiparameter interpolation.

The given work is devoted to the study of interpolation properties of Besov and Lizorkin–Triebel type spaces concerning interpolation method for anisotropic spaces (see in [8]).

In particular, the following theorem was proved.

Theorem. *Let* $-\infty < \sigma_0 \neq \sigma_1 < \infty$, $1 < p_0 \neq p_1 < \infty$, $1 < q_0, q_1 < \infty$ *and* $1 \le \tau_0, \tau_1 \le \infty$. *Then for* $0 < \theta_1, \theta_2 < 1$, $1 \le r_1, r_2 \le \infty$ *the following equalities are correct*:

a) *at* $\star_1 = (1,2)$

$$(B^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1} \tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star_1}} = B^{\sigma r_2}_{p r_1}(\mathbb{T}^n),$$

$$(F^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1} \tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star_1}} = B^{\sigma r_2}_{p r_1}(\mathbb{T}^n);$$

b) *at* $\star_2 = (2,1)$

$$(B^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1} \tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star_2}} = F^{\sigma r_2}_{p r_1}(\mathbb{T}^n),$$

$$(F^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1} \tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star_2}} = F^{\sigma r_2}_{p r_1}(\mathbb{T}^n);$$

where

$$\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1, \qquad 1/p = (1-\theta_1)/p_0 + \theta_1/p_1.$$

It is worth noting that, in contrast to the real valuable method, the scale of Besov and Lizorkin–Triebel type spaces is closed in the case of the method given by us; so, it behaves like a complete scale of spaces.

§1. Interpolation method for anisotropic spaces and interpolation of $l_q^\sigma(L_{p\tau})$ and $L_{p\tau}(\widehat{l_q^\sigma})$ spaces

Let $1 \le p, \tau \le \infty$, $\Omega \subset \mathbb{R}^n$. By Lorentz spaces $L_{pq}(\Omega)$ we call the set of measurable on Ω functions $f(x)$ for which the following norm is finite:

$$\|f\|_{L_{p\tau}(\Omega)} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^\tau \frac{dt}{t}\right)^{1/\tau} & \text{for } 1 \le p, \tau < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) & \text{for } 1 \le p \le \infty,\ \tau = \infty, \end{cases}$$

where $f^*(t)$ is the nonincreasing rearrangement of $f(x)$.

For $-\infty < \sigma < \infty$, $1 \le q, p, \tau \le \infty$, we define spaces $l_q^\sigma(L_{p\tau})$ and $L_{p\tau}(\widehat{l_q^\sigma})$, as the set of sequences $a = \{a_k(x)\}_{k=-\infty}^\infty$, where the $a_k(x)$ are functions, measurable on Ω, for which the following norms are respectively finite:

$$\|a\|_{l_q^\sigma(L_{p\tau})} = \Big(\sum_{k=-\infty}^{\infty} (2^{\sigma k}\|a_k\|_{L_{p\tau}})^q\Big)^{1/q},$$

$$\|a\|_{L_{p\tau}(\widehat{l_q^\sigma})} = \Big\|\Big(\sum_{k=-\infty}^{\infty} (2^{\sigma k} a_k^*(\cdot))^q\Big)^{1/q}\Big\|_{L_{p\tau}}.$$

L e m m a 1. *Let $-\infty < \sigma < \infty$ and $1 \le q, p, \tau \le \infty$. Then*

$$l_1^\sigma(L_{p\tau}) \hookrightarrow L_{p\tau}(\widehat{l_q^\sigma}) \hookrightarrow l_\infty^\sigma(L_{p\tau}). \tag{1}$$

P r o o f. Taking into account, that $q \ge 1$ and using the generalized Minkowski inequality, we obtain

$$\|a\|_{L_{p\tau}(\widehat{l_q^\sigma})} = \Big\|\Big(\sum_{k=0}^{\infty} (2^{\sigma k} a_k^*(\cdot))^q\Big)^{1/q}\Big\|_{L_{p\tau}} \le$$

$$\le \Big\|\sum_{k=0}^{\infty} 2^{\sigma k} a_k^*(\cdot)\Big\|_{L_{p\tau}} \le \sum_{k=0}^{\infty} 2^{\sigma k}\|a_k\|_{L_{p\tau}} = \|f\|_{l_1^\sigma(L_{p\tau})},$$

which means the following embedding

$$l_1^\sigma(L_{p\tau}) \hookrightarrow L_{p\tau}(\widehat{l_q^\sigma}). \tag{2}$$

Since $q \le \infty$, in a similar way we get

$$\|f\|_{L_{p\tau}(\widehat{l_q^\sigma})} = \Big\|\Big(\sum_{k=0}^{\infty} (2^{\sigma k} a_k^*(\cdot))^q\Big)^{1/q}\Big\|_{L_{p\tau}} \ge$$

$$\ge \Big\|\sup_{k\ge 0} 2^{\sigma k} a_k^*(\cdot)\Big\|_{L_{p\tau}} \ge \sup_{k\ge 0} 2^{\sigma k}\|a_k\|_{L_{p\tau}} = \|f\|_{l_\infty^\sigma(L_{p\tau})},$$

that is,

$$L_{p\tau}\Big(\widehat{l_q^\sigma}\Big) \hookrightarrow l_\infty^\sigma\Big(L_{p\tau}\Big). \tag{3}$$

Integrating (2) and (3), we obtain (1). Lemma 1 is proved.

In this paragraph we study the interpolation properties of $l_q^\sigma(L_{p\tau})$ and $L_{p\tau}(\widehat{l}_q^\sigma)$ spaces with respect to the interpolation method for anisotropic spaces (see [8]).

Let

$$E = \Big\{\varepsilon = (\varepsilon_1, \varepsilon_2) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1, i = 1, 2\Big\}$$

be a vertex of the unit square in $\mathbb{R}^2$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ a family of Banach spaces, which are subspaces of some linear Hausdorff's space called as a compatible family of Banach spaces. We define a functional for a from $\sum_{\varepsilon\in E} A_\varepsilon$ as follows:

$$K(\mathbf{t}, a; A_\varepsilon, \varepsilon \in E) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon}.$$

Let $\mathbf{0} < \theta = (\theta_1, \theta_2) < \mathbf{1}$, $\mathbf{0} < \mathbf{r} = (r_1, r_2) \le \infty$. For any rearrangement $\star = (j_1, j_2)$ of the set $\{1, 2\}$, we designate a linear subset of $\sum_{\varepsilon\in E} \mathbf{A}_\varepsilon$ by

$$\mathbf{A}_{\theta\mathbf{r}^\star} = (A_\varepsilon; \varepsilon \in E)_{\theta\mathbf{r}^\star},$$

for the elements of which the following norm is finite:

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}^\star}} =$$

$$= \Big(\int_0^\infty \Big(\int_0^\infty \Big(t_{j_1}^{-\theta_{j_1}} t_{j_2}^{-\theta_{j_2}} K(\mathbf{t}, a; A_\varepsilon, \varepsilon \in E)\Big)^{r_{j_1}} \frac{dt_{j_1}}{t_{j_1}}\Big)^{r_{j_2}/r_{j_1}} \frac{dt_{j_2}}{t_{j_2}}\Big)^{1/r_{j_2}} < \infty.$$

L e m m a 2 (see [8]). *Let* $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \le \infty$, $\star = (j_1, j_2)$ *be some rearrangement of the set* $\{1, 2\}$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$, $\mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ *two compatible families of Banach spaces. If there exist two vectors* $\mathbf{M_0} = (M_1^0, M_2^0)$, $\mathbf{M_1} = (M_1^1, M_2^1)$ *with positive components such that for the linear operator* $T : A_\varepsilon \to B_\varepsilon$ *with an estimate of the norm*

$$C_\varepsilon \prod_{i=1}^{2} M_i^{\varepsilon_i} \quad \text{for any } \varepsilon \in E,$$

is correct, then

$$T : \mathbf{A}_{\theta\mathbf{r}^\star} \to \mathbf{B}_{\theta\mathbf{r}^\star}$$

with the norm

$$\|T\|_{\mathbf{A}_{\theta\mathbf{r}^\star}\to\mathbf{B}_{\theta\mathbf{r}^\star}} \le \max_{\varepsilon\in E} C_\varepsilon \prod_{i=1}^{2} (M_i^0)^{1-\theta_i} (M_i^1)^{\theta_i}.$$

R e m a r k 1. We note that in case $\star = (1, 2)$ the given interpolation method was introduced in Fernandez [4].

T h e o r e m 1. *Let* $-\infty < \sigma_0 \neq \sigma_1 < +\infty$, $1 \le q_0, q_1 \le \infty$, $1 \le p_0 \neq p_1 \le \infty$ *and* $1 \le \tau_0, \tau_1 \le \infty$. *Then at* $0 < \theta_1, \theta_2 < 1$ *and* $1 \le r_1, r_2 \le \infty$ *the following equalities are correct*:

a) *if* $\star_1 = (1,2)$, *then*

$$(l_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_1}} = l_{r_2}^{\sigma}(L_{pr_1}),$$
$$(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(l_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_1}} = l_{r_2}^{\sigma}(L_{pr_1}),$$

b) *if* $\star_2 = (2,1)$, *then*

$$(l_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_2}} = L_{pr_1}(\widehat{l}_{r_2}^{\sigma}),$$
$$(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(l_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_2}} = L_{pr_1}(\widehat{l}_{r_2}^{\sigma}),$$

where

$$\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1, \qquad 1/p = (1-\theta_1)/p_0 + \theta_1/p_1.$$

P r o o f. Without loss of generality, we set $\sigma_0 > \sigma_1$ and $p_0 < p_1$. By the embeddings

$$l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1}) \hookrightarrow l_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}) \hookrightarrow l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty}), \qquad \varepsilon_1,\varepsilon_2 = 0,1,$$
$$l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1}) \hookrightarrow L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\widehat{l}_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}}) \hookrightarrow l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty}), \qquad \varepsilon_1,\varepsilon_2 = 0,1,$$

it is enough to prove that

$$(4) \qquad (l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_1}} \hookrightarrow$$
$$\hookrightarrow l_{r_2}^{\sigma}(L_{pr_1}) \hookrightarrow (l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_1}}$$

and

$$(5) \qquad (l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_2}} \hookrightarrow$$
$$\hookrightarrow L_{pr_1}(\widehat{l}_{r_2}^{\sigma}) \hookrightarrow (l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2 = 0,1)_{\theta\mathbf{r}^{\star_2}}.$$

Let

$$a = \{a_k(x)\}_{k\in\mathbb{Z}} \in \sum_{\varepsilon_1,\varepsilon_2=0}^{1} l_{q_{\varepsilon_2}}^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}), \qquad \text{where} \ \ \tau_{\varepsilon_1} = 1 \text{ or } \infty.$$

For each index $k \in \mathbb{Z}$ and $t_1 > 0$ we choose the set $E_{t_1}(k)$ so that

$$\mu(E_{t_1}(k)) = t_1^{\alpha} \quad \text{ and } \quad |a_k(x)| \ge a_k^*(t_1^{\alpha}) \ \text{ at } x \in E_{t_1}(k),$$

where $1/\alpha = 1/p_0 - 1/p_1$. Furthermore, for $t_2 > 0$ we choose the set of indexes

$$E_{t_2} = \{k : 2^{(\sigma_0-\sigma_1)k} \le t_2\}.$$

Let us assume that

$$a_k^{0,0}(x) = \begin{cases} a_k(x) & \text{if } \ k \in E_{t_2},\ x \in E_{t_1}(k), \\ 0 & \text{otherwise;} \end{cases}$$

$$a_k^{0,1}(x)=\begin{cases} a_k(x) & \text{if } k\in E_{t_2},\ x\notin E_{t_1}(k),\\ 0 & \text{otherwise;}\end{cases}$$

$$a_k^{1,0}(x)=\begin{cases} a_k(x) & \text{if } k\notin E_{t_2},\ x\in E_{t_1}(k)\\ 0 & \text{otherwise;}\end{cases}$$

$$a_k^{1,1}(x)=a_k(x)-(a_k^{0,0}(x)+a_k^{0,1}(x)+a_k^{1,0}(x)).$$

Then we have

$$K\Big(t_1,t_2,a;\ l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1}),\ \varepsilon_1,\varepsilon_2=0,1\Big)= \tag{6}$$

$$=\inf_{\substack{a=\sum_{\varepsilon_1,\varepsilon_2=0}^{1}a^{\varepsilon_1,\varepsilon_2}\\ a^{\varepsilon_1,\varepsilon_2}\in l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1})}}\sum_{\varepsilon_1,\varepsilon_2=0}^{1}t_1^{\varepsilon_1}t_2^{\varepsilon_2}\|a^{\varepsilon_1,\varepsilon_2}\|_{l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1})}\le$$

$$\le\sum_{k=1}^{\infty}\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})\Big(\int_0^{t_1^\alpha}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^\alpha}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)$$

and

$$K(t_1,t_2,a;l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty}),\ \varepsilon_1,\varepsilon_2=0,1)= \tag{7}$$

$$=\inf_{\substack{a=\sum_{\varepsilon_1,\varepsilon_2=0}^{1}a^{\varepsilon_1,\varepsilon_2}\\ a^{\varepsilon_1,\varepsilon_2}\in l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty})}}\sum_{\varepsilon_1,\varepsilon_2=0}^{1}t_1^{\varepsilon_1}t_2^{\varepsilon_2}\|a^{\varepsilon_1,\varepsilon_2}\|_{l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty})}\ge$$

$$\ge\sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})\Big(\sup_{0<s\le t_1^\alpha}s^{1/p_0}a_k^*(s)+t_1\sup_{s>t_1^\alpha}s^{1/p_1}a_k^*(s)\Big)\ge$$

$$\ge\sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})\sup_{0<s\le t_1^\alpha}s^{1/p_0}a_k^*(s).$$

Case (a). Let $\star_1=(1,2)$.

Step 1. According to estimate (3) and Minkowski's inequality, we obtain

$$\|a\|_{(l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}\star_1}}=\bigg(\int_0^\infty\bigg(\int_0^\infty\Big(t_1^{-\theta_1}t_2^{-\theta_2}\times \tag{8}$$

$$\times K(t_1,t_2,a;l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1}),\ \varepsilon_1,\varepsilon_2=0,1)\Big)^{r_1}\frac{dt_1}{t_1}\bigg)^{r_2/r_1}\frac{dt_2}{t_2}\bigg)^{1/r_2}\le$$

$$\le\bigg(\int_0^\infty\bigg(\int_0^\infty\Big(t_1^{-\theta_1}t_2^{-\theta_2}\sum_{k=1}^{\infty}\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})\times$$

$$\times\Big(\int_0^{t_1^\alpha}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^\alpha}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)\Big)^{r_1}\frac{dt_1}{t_1}\bigg)^{r_2/r_1}\frac{dt_2}{t_2}\bigg)^{1/r_2}\le$$

$$\le C_1\Big(\int_0^\infty \Big(t_2^{-\theta_2}\sum_{k=1}^\infty \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})\times$$

$$\times\Big(\int_0^\infty \Big(t_1^{-\theta_1}\int_0^{t_1^\alpha} s^{1/p_0}a_k^*(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+$$

$$+\Big(\int_0^\infty \Big(t_1^{1-\theta_1}\int_{t_1^\alpha}^\infty s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}=$$

$$=C_1\Big(\int_0^\infty \Big(t_2^{-\theta_2}\sum_{k\in\mathbb{Z}}\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})(I_1(k)+I_2(k))\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}.$$

We estimate $I_1(k)$ and $I_2(k)$ separately. By the replacement $s\to st_1^\alpha$, the generalized Minkowski's inequality and the inverse replacement $st_1^\alpha\to s$, we get

$$I_1(k)=\Big(\int_0^\infty \Big(t_1^{-\theta_1}\int_0^{t_1^\alpha} s^{1/p_0}a_k^*(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}= \tag{9}$$

$$=\Big(\int_0^\infty \Big(t_1^{-\theta_1+\alpha/p_0}\int_0^1 s^{1/p_0}a_k^*(st_1^\alpha)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\le$$

$$\le\int_0^1 s^{1/p_0}\Big(\int_0^\infty \Big(t_1^{-\theta_1+\alpha/p_0}a_k^*(st_1^\alpha)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s}=$$

$$=\int_0^1 s^{\theta_1/\alpha}\Big(\int_0^\infty \Big(t_1^{(1-\theta_1)/p_0+\theta_1/p_1}a_k^*(t_1)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s}=$$

$$=C_2\Big(\int_0^\infty (t_1^{1/p}a_k^*(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}=C_2\|a_k(\cdot)\|_{L_{pr_1}},$$

$$I_2(k)=\Big(\int_0^\infty \Big(t_1^{1-\theta_1}\int_{t_1^\alpha}^\infty s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}= \tag{10}$$

$$=\Big(\int_0^\infty \Big(t_1^{1-\theta_1+\alpha/p_0}\int_1^\infty s^{1/p_1}a_k^*(st_1^\alpha)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\le$$

$$\le\int_1^\infty s^{1/p_1}\Big(\int_0^\infty \Big(t_1^{1-\theta_1+\alpha/p_0}a_k^*(st_1^\alpha)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s}=$$

$$=\int_1^\infty s^{(\theta_1-1)/\alpha}\Big(\int_0^\infty \Big(t_1^{(1-\theta_1)/p_0+\theta/p_1}a_k^*(t_1)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s}=$$

$$=C_3\Big(\int_0^\infty (t_1^{1/p}a_k^*(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}=C_3\|a_k(\cdot)\|_{L_{pr_1}}.$$

If we substitute estimates (6) and (7) in (5), using the monotonicity of the functions $t_2^{-\theta_2}$, $\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})$ and Minkowski's inequality, we obtain

$$\|a\|_{(l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{1}}} \leq \tag{11}$$

$$\leq C_4\Big(\int_0^\infty \Big(t_2^{-\theta_2}\sum_{k=1}^\infty \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2} =$$

$$= C_4\Big(\sum_{j=-\infty}^{\infty}\int_{2^{(\sigma_0-\sigma_1)(j-1)}}^{2^{(\sigma_0-\sigma_1)j}} \Big(t_2^{-\theta_2}\sum_{k=1}^\infty \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2} \leq$$

$$\leq C_5\Big(\sum_{j=-\infty}^{\infty}\Big(2^{-\theta_2(\sigma_0-\sigma_1)j}\sum_{k=0}^\infty \min(2^{\sigma_0 k}, 2^{\sigma_1 k-(\sigma_1-\sigma_0)j})\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2} =$$

$$= C_5\Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=0}^\infty \min(2^{\sigma_0(k-j)}, 2^{\sigma_1(k-j)})\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2} =$$

$$= C_5\Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=-\infty}^{j} 2^{\sigma_0(k-j)}\|a_k(\cdot)\|_{L_{pr_1}} + $$

$$+ \sum_{k=j+1}^{+\infty} 2^{\sigma_1(k-j)}\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2} \leq$$

$$= C_5\Big(\Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=-\infty}^{j} 2^{\sigma_0(k-j)}\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2} +$$

$$+\Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=j+1}^{+\infty} 2^{\sigma_1(k-j)}\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2}\Big) = C_5(I_3+I_4).$$

To estimate I_3 and I_4, we choose κ_0 and κ_1 so that $\sigma_0 > \kappa_0 > \sigma > \kappa_1 > \sigma_1$. Then applying Hölder's inequality and changing the order of summation, we get

$$I_3 = \Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=-\infty}^{j} 2^{\sigma_0(k-j)}\Big\|a_k(\cdot)\Big\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2} = \tag{12}$$

$$= \Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_0)r_2 j}\Big(\sum_{k=-\infty}^{j} 2^{(\sigma_0-\kappa_0)k}2^{\kappa_0 k}\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2} \leq$$

$$\leq \Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_0)r_2 j}\Big(\sum_{k=-\infty}^{j} 2^{(\sigma_0-\kappa_0)kr'_2}\Big)^{r_2/r'_2}\sum_{k=-\infty}^{j}(2^{\kappa_0 k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\Big)^{1/r_2} \leq$$

$$\leq C_6\Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\kappa_0)r_2 j}\sum_{k=-\infty}^{j}(2^{\kappa_0 k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\Big)^{1/r_2} \leq$$

$$\le C_7\Big(\sum_{k=-\infty}^{\infty}(2^{\kappa_0 k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\sum_{j=k}^{+\infty}2^{(\sigma-\kappa_0)r_2 j}\Big)^{1/r_2}\le$$

$$\le C_8\Big(\sum_{k=-\infty}^{\infty}(2^{\sigma k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\Big)^{1/r_2}=C_8\|a\|_{l^{\sigma}_{r_2}(L_{pr_1})},$$

(13) $$I_4=\Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=j+1}^{+\infty}2^{\sigma_1(k-j)}\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2}=$$

$$=\Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_1)r_2 j}\Big(\sum_{k=j+1}^{+\infty}2^{(\sigma_1-\kappa_1)k}2^{\kappa_1 k}\|a_k(\cdot)\|_{L_{pr_1}}\Big)^{r_2}\Big)^{1/r_2}\le$$

$$\le\Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_1)r_2 j}\Big(\sum_{k=j+1}^{+\infty}2^{(\sigma_1-\kappa_1)kr'_2}\Big)^{r_2/r'_2}\times$$

$$\times\sum_{k=j+1}^{+\infty}(2^{\kappa_1 k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\Big)^{1/r_2}\le$$

$$\le C_9\Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\kappa_1)r_2 j}\sum_{k=j+1}^{+\infty}(2^{\kappa_1 k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\Big)^{1/r_2}\le$$

$$\le C_{10}\Big(\sum_{k=-\infty}^{\infty}(2^{\kappa_1 k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\sum_{j=-\infty}^{k-1}2^{(\sigma-\kappa_1)r_2 j}\Big)^{1/r_2}\le$$

$$\le C_{11}\Big(\sum_{k=-\infty}^{\infty}(2^{\sigma k}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\Big)^{1/r_2}=C_{11}\|a\|_{l^{\sigma}_{r_2}(L_{pr_1})}.$$

By substituting estimates (12) and (13) in (11), we get

$$\|a\|_{(l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{1}}}\le C_{12}\|a\|_{l^{\sigma}_{r_2}(L_{pr_1})},$$

which means that

(14) $$l^{\sigma}_{r_2}(L_{pr_1})\hookrightarrow(l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{1}}.$$

Step 2. Using estimate (7), the replacement $s\to st_1^{\alpha}$, the generalized Minkowski inequality, the inverse replacement $st_1^{\alpha}\to s$, and also the monotonicity of the functions $t_2^{-\theta_2}$, $\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})$, we obtain

$$\|a\|_{(l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty});\varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{1}}}=\Big(\int_0^\infty\Big(\int_0^\infty\Big(t_1^{-\theta_1}t_2^{-\theta_2}\times$$

$$\times K(t_1,t_2,a;l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty}),\ \varepsilon_1,\varepsilon_2=0,1)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{r_2/r_1}\frac{dt_2}{t_2}\Big)^{1/r_2}\ge$$

$$\geq \Big(\int_0^\infty \Big(\int_0^\infty \Big(t_1^{-\theta_1} t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \times$$

$$\times \Big(\sup_{0<s\leq t_1^\alpha} s^{1/p_0} a_k^*(s) \Big)\Big)^{r_1} \frac{dt_1}{t_1} \Big)^{r_2/r_1} \frac{dt_2}{t_2} \Big)^{1/r_2} =$$

$$\geq \Big(\int_0^\infty \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \times$$

$$\times \Big(\int_0^\infty \Big(t_1^{-\theta_1} \sup_{0<s\leq t_1^\alpha} s^{1/p_0} a_k^*(s) \Big)^{r_1} \frac{dt_1}{t_1} \Big)^{1/r_1} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} \geq$$

$$\geq \Big(\int_0^\infty \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \times$$

$$\times \Big(\int_0^\infty (t_1^{-\theta_1+\alpha/p_0} \sup_{0<s\leq 1} s^{1/p_0} a_k^*(s t_1^\alpha) \Big)^{r_1} \frac{dt_1}{t_1} \Big)^{1/r_1} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} \geq$$

$$\geq \Big(\int_0^\infty \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \sup_{0<s\leq 1} s^{1/p_0} \times$$

$$\times \Big(\int_0^\infty (t_1^{-\theta_1+\alpha/p_0} a_k^*(s t_1^\alpha))^{r_1} \frac{dt_1}{t_1} \Big)^{1/r_1})^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} =$$

$$= \Big(\int_0^\infty \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \sup_{0<s\leq 1} s^{\theta_1/\alpha} \times$$

$$\times \Big(\int_0^\infty (t_1^{(1-\theta_1)/p_0+\theta_1/p_1} a_k^*(t_1))^{r_1} \frac{dt_1}{t_1} \Big)^{1/r_1} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} =$$

$$= \Big(\int_0^\infty \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \times$$

$$\times \Big(\int_0^\infty (t_1^{1/p} a_k^*(t_1))^{r_1} \frac{dt_1}{t_1} \Big)^{1/r_1} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} =$$

$$= \Big(\int_0^\infty \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \|a_k(\cdot)\|_{L_{pr_1}} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} =$$

$$= \Big(\sum_{j=-\infty}^{\infty} \int_{2^{(\sigma_0-\sigma_1)(j-1)}}^{2^{(\sigma_0-\sigma_1)j}} \Big(t_2^{-\theta_2} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) \|a_k(\cdot)\|_{L_{pr_1}} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} \geq$$

$$\geq C_{13} \Big(\sum_{j=-\infty}^{\infty} \Big(2^{-\theta_2(\sigma_0-\sigma_1)j} \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0 k}, 2^{\sigma_1 k+(\sigma_0-\sigma_1)j}) \times$$

$$\times \|a_k(\cdot)\|_{L_{pr_1}} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} \geq C_{13} \Big(\sum_{j=-\infty}^{\infty} \Big(2^{((1-\theta_2)\sigma_0+\theta_2\sigma_1)j} \times$$

$$\times \sup_{k\in\mathbb{Z}} \min(2^{\sigma_0(k-j)}, 2^{\sigma_1(k-j)}) \|a_k(\cdot)\|_{L_{pr_1}} \Big)^{r_2} \frac{dt_2}{t_2} \Big)^{1/r_2} \geq$$

$$\geq C_{13}\Big(\sum_{j=-\infty}^{\infty}(2^{\sigma j}\|a_k(\cdot)\|_{L_{pr_1}})^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}=C_{13}\|a\|_{l^{\sigma}_{r_2}(L_{pr_1})},$$

which means that

$$(l^{\sigma_{\varepsilon_2}}_{\infty}(L_{p_{\varepsilon_1}\infty});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}\star\mathbf{1}}\hookrightarrow l^{\sigma}_{r_2}(L_{pr_1}). \tag{15}$$

Step 3. Combining (14) and (15) completes the proof of Case (a).

Case (b) Let $\star_2=(2,1)$.

Step 4. Using estimate (6), monotonicity of the functions $t_2^{-\theta_2}$, $\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})$, and the Minkowski inequality, we get

$$\|a\|_{(l^{\sigma_{\varepsilon_2}}_{1}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}\star\mathbf{2}}}=\Big(\int_0^{\infty}\Big(\int_0^{\infty}\Big(t_1^{-\theta_1}t_2^{-\theta_2}\times \tag{16}$$

$$\times K(t_1,t_2,a;l^{\sigma_{\varepsilon_2}}_1(L_{p_{\varepsilon_1}1}),\,\varepsilon_1,\varepsilon_2=0,1)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{r_1/r_2}\frac{dt_1}{t_1}\Big)^{1/r_1}\leq$$

$$\leq\Big(\int_0^{\infty}\Big(\int_0^{\infty}\Big(t_1^{-\theta_1}t_2^{-\theta_2}\sum_{k=1}^{\infty}\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})\times$$

$$\times\Big(\int_0^{t_1^{\alpha}}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^{\alpha}}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{r_1/r_2}\frac{dt_1}{t_1}\Big)^{1/r_1}=$$

$$=\Big(\int_0^{\infty}\Big(t_1^{-\theta_1}\Big(\sum_{j=-\infty}^{\infty}\int_{2^{(\sigma_0-\sigma_1)j}}^{2^{b(\sigma_0-\sigma_1)(j+1)}}\Big(t_2^{-\theta_2}\sum_{k=1}^{\infty}\min(2^{\sigma_0 k},t_2 2^{\sigma_1 k})\times$$

$$\times\Big(\int_0^{t_1^{\alpha}}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^{\alpha}}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\leq$$

$$\leq C_{14}\Big(\int_0^{\infty}\Big(t_1^{-\theta_1}\Big(\sum_{j=-\infty}^{\infty}\Big(2^{-\theta_2(\sigma_0-\sigma_1)j}\sum_{k=1}^{\infty}\min(2^{\sigma_0 k},2^{\sigma_1 k-(\sigma_1-\sigma_0)j})\times$$

$$\times\Big(\int_0^{t_1^{\alpha}}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^{\alpha}}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)\Big)^{r_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}=$$

$$=C_{14}\Big(\int_0^{\infty}\Big(t_1^{-\theta_1}\Big(\sum_{j=-\infty}^{\infty}\Big(2^{((1-\theta_2)\sigma_0+\theta_2\sigma_1)j}\sum_{k=1}^{\infty}\min(2^{\sigma_0(k-j)},2^{\sigma_1(k-j)})\times$$

$$\times\Big(\int_0^{t_1^{\alpha}}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^{\alpha}}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)\Big)^{r_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}=$$

$$=C_{14}\Big(\int_0^{\infty}\Big(t_1^{-\theta_1}\Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=1}^{\infty}\min(2^{\sigma_0(k-j)},2^{\sigma_1(k-j)})\times$$

$$\times\Big(\int_0^{t_1^{\alpha}}s^{1/p_0}a_k^*(s)\frac{ds}{s}+t_1\int_{t_1^{\alpha}}^{\infty}s^{1/p_1}a_k^*(s)\frac{ds}{s}\Big)\Big)^{r_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\leq$$

$$\leq C_{14}\Big(\int_0^\infty\Big(t_1^{-\theta_1}\Big(\int_0^{t_1^\alpha}s^{1/p_0}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\sum_{k=1}^\infty\min(2^{\sigma_0(k-j)},2^{\sigma_1(k-j)})\times$$

$$\times a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}+t_1\int_{t_1^\alpha}^\infty s^{1/p_1}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\sum_{k=1}^\infty\min(2^{\sigma_0(k-j)},2^{\sigma_1(k-j)})\times$$

$$\times a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}\Big)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}=C_{14}\Big(\int_0^\infty\Big(t_1^{-\theta_1}\Big(\int_0^{t_1^\alpha}s^{1/p_0}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\times$$

$$\times\Big(\sum_{k=-\infty}^{j}2^{\sigma_0(k-j)}a_k^*(s)+\sum_{k=j+1}^\infty 2^{\sigma_1(k-j)}a_k^*(s)\Big)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}+$$

$$+t_1\int_{t_1^\alpha}^\infty s^{1/p_1}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\Big(\sum_{k=-\infty}^{j}2^{\sigma_0(k-j)}a_k^*(s)+$$

$$+\sum_{k=j+1}^\infty 2^{\sigma_1(k-j)}a_k^*(s)\Big)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}\Big)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\leq$$

$$\leq C_{15}\Big(\Big(\int_0^\infty\Big(t_1^{-\theta_1}\int_0^{t_1^\alpha}s^{1/p_0}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\sum_{k=-\infty}^{j}2^{\sigma_0(k-j)}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\times$$

$$\times\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+\Big(\int_0^\infty\Big(t_1^{1-\theta_1}\int_0^{t_1^\alpha}s^{1/p_0}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\times$$

$$\times\sum_{k=j+1}^\infty 2^{\sigma_1(k-j)}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+\Big(\int_0^\infty\Big(t_1^{-\theta_1}\int_{t_1^\alpha}^\infty s^{1/p_1}\times$$

$$\times\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\sum_{k=-\infty}^{j}2^{\sigma_0(k-j)}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+\Big(\int_0^\infty\Big(t_1^{1-\theta_1}\times$$

$$\times\int_{t_1^\alpha}^\infty s^{1/p_1}\Big(\sum_{j=-\infty}^\infty\Big(2^{\sigma j}\sum_{k=j+1}^\infty 2^{\sigma_1(k-j)}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\Big)=$$

$$=C_{15}\Big(\Big(\int_0^\infty\Big(t_1^{-\theta_1}\int_0^{t_1^\alpha}s^{1/p_0}I_5(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+$$

$$+\Big(\int_0^\infty\Big(t_1^{-\theta_1}\int_0^{t_1^\alpha}s^{1/p_0}I_6(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+$$

$$+\Big(\int_0^\infty\Big(t_1^{1-\theta_1}\int_{t_1^\alpha}^\infty s^{1/p_1}I_5(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}+$$

$$+\Big(\int_0^\infty\Big(t_1^{1-\theta_1}\int_{t_1^\alpha}^\infty s^{1/p_1}I_6(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\Big).$$

To estimate $I_5(s)$ and $I_6(s)$, as well as in Step 1, we choose κ_0 and κ_1 so that

$$\sigma_0 > \kappa_0 > \sigma > \kappa_1 > \sigma_1.$$

Then applying Hölder's inequality and changing the order of summation gives

$$I_3(s) = \Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=-\infty}^{j}2^{\sigma_0(k-j)}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2} = \tag{17}$$

$$= \Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_0)r_2 j}\Big(\sum_{k=-\infty}^{j}2^{(\sigma_0-\kappa_0)k}2^{\kappa_0 k}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2} \le$$

$$\le \Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_0)r_2 j}\Big(\sum_{k=-\infty}^{j}2^{(\sigma_0-\kappa_0)kr'_2}\Big)^{r_2/r'_2}\sum_{k=-\infty}^{j}(2^{\kappa_0 k}a_k^*(s))^{r_2}\Big)^{1/r_2} \le$$

$$\le C_{16}\Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\kappa_0)r_2 j}\sum_{k=-\infty}^{j}(2^{\kappa_0 k}a_k^*(s))^{r_2}\Big)^{1/r_2} \le$$

$$\le C_{17}\Big(\sum_{k=-\infty}^{\infty}(2^{\kappa_0 k}a_k^*(s))^{r_2}\sum_{j=k}^{\infty}2^{(\sigma-\kappa_0)r_2 j}\Big)^{1/r_2} \le$$

$$\le C_{18}\Big(\sum_{k=-\infty}^{\infty}(2^{\sigma k}a_k^*(s))^{r_2}\Big)^{1/r_2} = C_{18}f_{\sigma,r_2}(s),$$

$$I_4(s) = \Big(\sum_{j=-\infty}^{\infty}\Big(2^{\sigma j}\sum_{k=j+1}^{\infty}2^{\sigma_1(k-j)}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2} = \tag{18}$$

$$= \Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_1)r_2 j}\Big(\sum_{k=j+1}^{\infty}2^{(\sigma_1-\kappa_1)k}2^{\kappa_1 k}a_k^*(s)\Big)^{r_2}\Big)^{1/r_2} \le$$

$$\le \Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\sigma_1)r_2 j}\Big(\sum_{k=j+1}^{\infty}2^{(\sigma_1-\kappa_1)kr'_2}\Big)^{r_2/r'_2}\sum_{k=j+1}^{\infty}(2^{\kappa_1 k}a_k^*(s))^{r_2}\Big)^{1/r_2} \le$$

$$\le C_{19}\Big(\sum_{j=-\infty}^{\infty}2^{(\sigma-\kappa_1)r_2 j}\sum_{k=j+1}^{\infty}(2^{\kappa_1 k}a_k^*(s))^{r_2}\Big)^{1/r_2} \le$$

$$\le C_{20}\Big(\sum_{j=-\infty}^{\infty}(2^{\kappa_1 k}a_k^*(s))^{r_2}\sum_{j=-\infty}^{k-1}2^{(\sigma-\kappa_1)r_2 j}\Big)^{1/r_2} \le$$

$$\le C_{21}\Big(\sum_{j=-\infty}^{\infty}(2^{\sigma k}a_k^*(s))^{r_2}\Big)^{1/r_2} = C_{21}f_{\sigma,r_2}(s).$$

Substituting estimates (17) and (18) in (16) yields

$$\|a\|_{(l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\,\varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^{\star}\mathbf{2}}} \leq \tag{19}$$

$$\leq C_{22}\Big(\Big(\int_0^\infty \Big(t_1^{-\theta_1}\int_0^{t_1^\alpha} s^{1/p_0} f_{\sigma,r_2}(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} +$$

$$+\Big(\int_0^\infty \Big(t_1^{1-\theta_1}\int_{t_1^\alpha}^{\infty} s^{1/p_1} f_{\sigma,r_2}(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\Big) = I_7 + I_8.$$

Let's estimate I_7 and I_8. Applying the replacement $s \to st_1^\alpha$, the generalized Minkowski inequality and the inverse replacement $st_1^\alpha \to s$, we obtain

$$I_7 = \Big(\int_0^\infty \Big(t_1^{-\theta_1}\int_0^{t_1^\alpha} s^{1/p_0} f_{\sigma,r_2}(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} = \tag{20}$$

$$= \Big(\int_0^\infty \Big(t_1^{-\theta_1+\alpha/p_0}\int_0^{1} s^{1/p_0} f_{\sigma,r_2}(st_1^\alpha)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \leq$$

$$\leq \int_0^1 s^{1/p_0}\Big(\int_0^\infty (t_1^{-\theta_1+\alpha/p_0} f_{\sigma,r_2}(st_1^\alpha))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s} =$$

$$= \int_0^1 s^{\theta_1/\alpha}\Big(\int_0^\infty (t_1^{(1-\theta_1)/p_0+\theta_1/p_1} f_{\sigma,r_2}(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s} =$$

$$= C_{23}\Big(\int_0^\infty (t_1^{1/p} f_{\sigma,r_2}(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} = C_{23}\|f_{\sigma,r_2}(\cdot)\|_{L_{pr_1}} =$$

$$= C_{23}\Big\|\Big(\sum_{j=-\infty}^{\infty}(2^{\sigma k}a_k^*(\cdot))^{r_2}\Big)^{1/r_2}\Big\|_{L_{pr_1}} = C_{23}\|a\|_{L_{pr_1}(\widehat{l}_{r_2}^{\sigma})},$$

$$I_8 = \Big(\int_0^\infty \Big(t_1^{1-\theta_1}\int_{t_1^\alpha}^{\infty} s^{1/p_1} f_{\sigma,r_2}(s)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} = \tag{21}$$

$$= \Big(\int_0^\infty \Big(t_1^{1-\theta_1+\alpha/p_1}\int_1^{\infty} s^{1/p_1} f_{\sigma,r_2}(st_1^\alpha)\frac{ds}{s}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \leq$$

$$\leq \int_1^\infty s^{1/p_1}\Big(\int_0^\infty (t_1^{1-\theta_1+\alpha/p_1} f_{\sigma,r_2}(st_1^\alpha))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s} =$$

$$= \int_1^\infty s^{(\theta_1-1)/\alpha}\Big(\int_0^\infty (t_1^{(1-\theta_1)/p_0+\theta_1/p_1} f_{\sigma,r_2}(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}\frac{ds}{s} =$$

$$= C_{24}\Big(\int_0^\infty (t_1^{1/p} f_{\sigma,r_2}(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} = C_{24}\|f_{\sigma,r_2}(\cdot)\|_{L_{pr_1}} =$$

$$= C_{24}\Big\|\Big(\sum_{j=-\infty}^{\infty}(2^{\sigma k}a_k^*(\cdot))^{r_2}\Big)^{1/r_2}\Big\|_{L_{pr_1}} = C_{24}\|a\|_{L_{pr_1}(\widehat{l}_{r_2}^{\sigma})}.$$

Substituting estimates (20) and (21) in (19), we get

$$\|a\|_{(l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{2}}} \le C_{25}\|a\|_{L_{pr_1}(\widehat{l}^{\sigma}_{r_2})},$$

which means that

$$L_{pr_1}(\widehat{l}^{\sigma}_{r_2}) \hookrightarrow (l_1^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}1});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{2}}. \tag{22}$$

Step 5. Using estimate (7), and also the monotonicity of the functions $t_2^{-\theta_2}$, $\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})$, we obtain

$$\|a\|_{(l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty});\,\varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{2}}} = \Big(\int_0^\infty \Big(\int_0^\infty \Big(t_1^{-\theta_1} t_2^{-\theta_2} \times$$

$$\times K(t_1,t_2,a; l_\infty^{\sigma_{\varepsilon_2}}(L_{p_{\varepsilon_1}\infty}),\,\varepsilon_1,\varepsilon_2=0,1)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{r_1/r_2}\frac{dt_1}{t_1}\Big)^{1/r_1} \ge$$

$$\ge \Big(\int_0^\infty \Big(\int_0^\infty \Big(t_1^{-\theta_1} t_2^{-\theta_2}\Big(\sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})\sup_{0<s\le t_1^\alpha} s^{1/p_0} a_k^*(s)\Big)\Big)^{r_2} \times$$

$$\times \frac{dt_2}{t_2}\Big)^{r_1/r_2}\frac{dt_1}{t_1}\Big)^{1/r_1} = \Big(\int_0^\infty \Big(t_1^{-\theta_1}\Big(\int_0^\infty \Big(t_2^{-\theta_2} \times$$

$$\times \sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k})\sup_{0<s\le t_1^\alpha} s^{1/p_0} a_k^*(s)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \ge$$

$$\ge \Big(\int_0^\infty \Big(t_1^{-\theta_1}\sup_{0<s\le t_1^\alpha} s^{1/p_0}\Big(\int_0^\infty \Big(t_2^{-\theta_2} \times$$

$$\times \sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) a_k^*(s)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} =$$

$$= \Big(\int_0^\infty \Big(t_1^{-\theta_1}\sup_{0<s\le t_1^\alpha} s^{1/p_0}\Big(\sum_{j\in\mathbb{Z}}\int_{2^{(\sigma_0-\sigma_1)(j-1)}}^{2^{(\sigma_0-\sigma_1)j}} \Big(t_2^{-\theta_2} \times$$

$$\times \sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k}, t_2 2^{\sigma_1 k}) a_k^*(s)\Big)^{r_2}\frac{dt_2}{t_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \ge$$

$$\ge \Big(\int_0^\infty \Big(t_1^{-\theta_1}\sup_{0<s\le t_1^\alpha} s^{1/p_0}\Big(\sum_{j\in\mathbb{Z}}\Big(2^{-(\sigma_0-\sigma_1)\theta_2 j} \times$$

$$\times \sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 k}, 2^{(\sigma_0-\sigma_1)j+\sigma_1 k}) a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} =$$

$$= \Big(\int_0^\infty \Big(t_1^{-\theta_1}\sup_{0<s\le t_1^\alpha} s^{1/p_0}\Big(\sum_{j\in\mathbb{Z}}\Big(2^{((1-\theta_2)\sigma_0+\theta_2\sigma_1)j} \times$$

$$\times \sup_{k\in\mathbb{Z}}\min(2^{\sigma_0 (k-j)}, 2^{\sigma_1 (k-j)}) a_k^*(s)\Big)^{r_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \ge$$

$$\geq \Big(\int_0^\infty \Big(t_1^{-\theta_1} \sup_{0<s\leq t_1^\alpha} s^{1/p_0}\Big(\sum_{j\in\mathbb{Z}}(2^{\sigma j}a_j^\star(s))^{r_2}\Big)^{1/r_2}\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} =$$

$$= \Big(\int_0^\infty \Big(t_1^{-\theta_1} \sup_{0<s\leq t_1^\alpha} s^{1/p_0} f_{\sigma,r_2}(s)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1}.$$

By the replacement $s \to st_1^\alpha$, the generalized Minkowski inequality and the inverse replacement $st_1^\alpha \to s$, we get

$$\|a\|_{(l_\infty^{\sigma\varepsilon_2}(L_{p_{\varepsilon_1}\infty}); \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{2}}} \geq$$

$$\geq \Big(\int_0^\infty \Big(t_1^{-\theta_1} \sup_{0<s\leq t_1^\alpha} s^{1/p_0} f_{\sigma,r_2}(s)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \geq$$

$$\geq \Big(\int_0^\infty \Big(t_1^{-\theta_1+\alpha/p_0} \sup_{0<s\leq 1} s^{1/p_0} f_{\sigma,r_2}(st_1^\alpha)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \geq$$

$$\geq \sup_{0<s\leq 1} s^{1/p_0}\Big(\int_0^\infty \Big(t_1^{-\theta_1} f_{\sigma,r_2}(st_1^\alpha)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} \geq$$

$$\geq \sup_{0<s\leq 1} s^{\theta_1/\alpha}\Big(\int_0^\infty (t_1^{-\theta_1/\alpha+1/p_0} f_{\sigma,r_2}(t_1))^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} =$$

$$= \Big(\int_0^\infty \Big(t_1^{(1-\theta_1)/p_0+\theta_1/p_1} f_{\sigma,r_2}(t_1)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} =$$

$$= \Big(\int_0^\infty \Big(t_1^{1/p} f_{\sigma,r_2}(t_1)\Big)^{r_1}\frac{dt_1}{t_1}\Big)^{1/r_1} =$$

$$= \Big\|\Big(\sum_{j=-\infty}^{\infty}(2^{\sigma k}a_k^*(\cdot))^{r_2}\Big)^{1/r_2}\Big\|_{L_{pr_1}} = \|a\|_{L_{pr_1}\left(\widehat{l}_{r_2}^\sigma\right)},$$

which means that

$$(l_1^{\sigma\varepsilon_2}(L_{p_{\varepsilon_1}\infty});\ \varepsilon_1,\varepsilon_2=0,1)_{\theta\mathbf{r}^\star\mathbf{2}} \hookrightarrow L_{pr_1}(\widehat{l}_{r_2}^\sigma). \qquad (23)$$

Step 6. Combining (22) and (23) completes the proof of Case (b).

Theorem 1 is proved.

§2. Spaces $B_{p\tau}^{\sigma q}$ and $F_{p\tau}^{\sigma q}$ and their interpolation

Let $1 \le p, \tau \le \infty$. For the function $f \in L_{p\tau}(\mathbb{T}^n)$ let us define

$$\Delta_k(f, \mathbf{x}) = \sum_{\mathbf{j} \in \rho(k)} a_{\mathbf{j}}(f) e^{i\langle \mathbf{j}, \mathbf{x}\rangle},$$

where the $\{a_{\mathbf{j}}(f) : \mathbf{j} \in \mathbb{Z}^n\}$ are the Fourier coefficients of f with respect to the multiple trigonometric system,

$$\langle \mathbf{y}, \mathbf{x}\rangle = \sum_{i=1}^{n} y_i x_i, \quad \rho(k) = \Big\{\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n : \ \mathbf{j} \in Q_k \setminus Q_{k-1}\Big\},$$

$$Q_0 = (0, \dots, 0),$$

$$Q_k = \Big\{\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n : \ |j_i| < 2^k, \ i = 1, \dots, n\Big\}, \quad k \in \mathbb{N}.$$

Let $-\infty < \sigma < \infty$ and $1 \le q \le \infty$. The set of functions f from $L_{p\tau}(\mathbb{T}^n)$ for which the following norms

$$\|f\|_{B_{p\tau}^{\sigma q}(\mathbb{T}^n)} = \|\Delta_k(f, \cdot)\|_{l_q^\sigma(L_{p\tau})(\mathbb{T}^n)} \quad \text{and} \quad \|f\|_{F_{p\tau}^{\sigma q}(\mathbb{T}^n)} = \|\Delta_k(f, \cdot)\|_{L_{p\tau}(\widehat{l_q^\sigma})(\mathbb{T}^n)}$$

are finite, are called as Besov type spaces $B_{p\tau}^{\sigma q}(\mathbb{T}^n)$ and Lizorkin–Triebel type spaces $F_{p\tau}^{\sigma q}(\mathbb{T}^n)$, respectively.

R e m a r k 2. The spaces $B_{p\tau}^{\sigma q}(\mathbb{T}^n)$ are the periodic analogues of the spaces $B_{p,q,(\tau)}^{\sigma}(\mathbb{R}^n)$ (see [3], [10]), and the spaces $F_{p\tau}^{\sigma q}(\mathbb{T}^n)$ by definition are somewhat similar to the periodic analogues of the spaces $F_{p,q,(\tau)}^{\sigma}(\mathbb{R}^n)$ (see [10]). However, in contrast to these cases, nonincreasing rearragement is applied in our case to each function $\Delta_k(f, x)$ in place of

$$f_{\sigma,q}(x) = \Big(\sum_{k=0}^{\infty} (2^{\sigma k} |\Delta_k(f, x)|)^q\Big)^{1/q}.$$

L e m m a 3. *Let $-\infty < \sigma < \infty$, $1 \le q \le \infty$ and $1 \le p, \tau \le \infty$. Then*

$$B_{p\tau}^{\sigma 1}(\mathbb{T}^n) \hookrightarrow F_{p\tau}^{\sigma q}(\mathbb{T}^n) \hookrightarrow B_{p\tau}^{\sigma \infty}(\mathbb{T}^n).$$

P r o o f. It follows from Lemma 1.

T h e o r e m 2. *Let $-\infty < \sigma_0 \ne \sigma_1 < \infty$, $1 \le p_0 \ne p_1 \le \infty$, $1 \le q_0, q_1, \tau_0, \tau_1 \le \infty$. Then for $0 < \theta_1, \theta_2 < 1$, $1 \le r_1, r_2 \le \infty$ the following equalities are correct*:

(a) *at* $\star_1 = (1, 2)$

$$(B_{p_{\varepsilon_1}\tau_{\varepsilon_1}}^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0, 1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star_1}} = B_{pr_1}^{\sigma r_2}(\mathbb{T}^n),$$

$$(F_{p_{\varepsilon_1}\tau_{\varepsilon_1}}^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0, 1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star_1}} = B_{pr_1}^{\sigma r_2}(\mathbb{T}^n),$$

(b) *at* $\star_2 = (2,1)$

$$(B^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 2}} = F^{\sigma r_2}_{pr_1}(\mathbb{T}^n),$$

$$(F^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 2}} = F^{\sigma r_2}_{pr_1}(\mathbb{T}^n),$$

where

$$\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1, \qquad 1/p = (1-\theta_1)/p_0 + \theta_1/p_1.$$

P r o o f. (a) Let's prove the first equality. According to Theorem 1(a), we have

$$\|f\|_{(B^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 1}}} =$$
$$= \|\{\Delta_k(f)\}\|_{(l^{\sigma_{\varepsilon_2}}_{q_{\varepsilon_2}}(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}})(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 1}}} \sim$$
$$\sim \|\{\Delta_k(f)\}\|_{l^{\sigma}_{r_2}(L_{pr_1})(\mathbb{T}^n)} = \|f\|_{B^{\sigma r_2}_{pr_1}(\mathbb{T}^n)},$$

where $\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1$, $1/p = (1-\theta_1)/p_0 + \theta_1/p_1$.

Let's prove the second equality. Similarly, using the second equality of Theorem 1(a), we get

$$\|f\|_{(F^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 1}}} =$$
$$= \|\{\Delta_k(f)\}\|_{(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\widehat{l}^{\sigma_{\varepsilon_2}}_{q_{\varepsilon_2}})(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 1}}} \sim$$
$$\sim \|\{\Delta_k(f)\}\|_{l^{\sigma}_{r_2}(L_{pr_1})(\mathbb{T}^n)} = \|f\|_{B^{\sigma r_2}_{pr_1}(\mathbb{T}^n)},$$

where

$$\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1, \quad 1/p = (1-\theta_1)/p_0 + \theta_1/p_1.$$

(b) Let's prove the first equality. According to Theorem 1(b), we have

$$\|f\|_{(B^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 2}}} =$$
$$= \|\{\Delta_k(f)\}\|_{(l^{\sigma_{\varepsilon_2}}_{q_{\varepsilon_2}}(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n)); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 2}}} \sim$$
$$\sim \|\{\Delta_k(f)\}\|_{L_{pr_1}(\widehat{l}^{\sigma}_{r_2})(\mathbb{T}^n)} = \|f\|_{F^{\sigma r_2}_{pr_1}(\mathbb{T}^n)},$$

where

$$\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1, \quad 1/p = (1-\theta_1)/p_0 + \theta_1/p_1.$$

Let's prove the second equality. Using the second equality of Theorem 1(b), we get

$$\|f\|_{(F^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 1}}} =$$
$$= \|\{\Delta_k(f)\}\|_{(L_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\widehat{l}^{\sigma_{\varepsilon_2}}_{q_{\varepsilon_2}})(\mathbb{T}^n); \varepsilon_1, \varepsilon_2 = 0,1)_{(\theta_1,\theta_2),(r_1,r_2)^{\star 2}}} \sim$$
$$\sim \|\{\Delta_k(f)\}\|_{L_{pr_1}(\widehat{l}^{\sigma}_{r_2})(\mathbb{T}^n)} = \|f\|_{F^{\sigma r_2}_{pr_1}(\mathbb{T}^n)},$$

where

$$\sigma = (1-\theta_2)\sigma_0 + \theta_2\sigma_1, \quad 1/p = (1-\theta_1)/p_0 + \theta_1/p_1.$$

Theorem 2 is proved.

Remark 3. Similarly, it is possible to consider $B^{\sigma q}_{p\tau}(\mathbb{R}^n)$ and $F^{\sigma q}_{p\tau}(\mathbb{R}^n)$ spaces. For this it is necessary to consider a sequence $\{F^{-1}\varphi_k Ff(x)\}$ instead of the sequence $\{\Delta_k(f,x)\}$, where F, F^{-1} are the direct and inverse Fourier transforms, respectively, and the sequence $\varphi = \{\varphi_k(x)\}_{k\in\mathbb{Z}_+}$ satisfies the following condition (see [3]):

$$\operatorname{supp}\varphi_0 \subset \{x : |x| \le 2\}, \quad \operatorname{supp}\varphi_k \subset \{x : 2^{k-1} \le |x| \le 2^{k+1}\},\ k = 1,2,\dots;$$

for each multiindex $\alpha = (\alpha_1,\dots,\alpha_n)$ there exists a positive number c_α, such that

$$2^{k|\alpha|}\Big|\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,\varphi_k(x)\Big| \le c_\alpha \qquad \text{for all}\ \ k = 1,2,\dots\ \text{and all}\ x\in\mathbb{R}^n,$$

$$\sum_{k=0}^{\infty}\varphi_k(x) = 1 \qquad \text{for all}\ \ x\in\mathbb{R}^n.$$

For the given spaces it is also possible to prove an anologue of Theorem 2.

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Интерполяция пространств Бесова $B^{\sigma q}_{p\tau}$ и Лизоркина–Трибеля $F^{\sigma q}_{p\tau}$

КУАНЫШ БЕКМАГАНБЕТОВ и ЕРЛАН НУРСУЛТАНОВ

В статье рассматривается интерполяционный метод для анизотропных пространств, обобщаюший метод Фернандеса. Исследуются интерполяционные свойства пространств Бесова $B^{\sigma q}_{p\tau}$ и Лизоркина–Трибеля $F^{\sigma q}_{p\tau}$ относительно быше указанного метода. Показана замкнутость шкал этих пространств относительно рассмотренного интерполяционного метода.