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Lower and Upper Bounds for the Norm of Multipliers of Multpile Trigonometric Fourier Series in Lebesgue Spaces

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Let $1 < p \leqslant q < \infty$, and let $\mathbb{T}^n$ be the n-dimensional torus. We say that a sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}^n}$ of complex numbers is a *multiplier of trigonometric Fourier series* from $L_p(\mathbb{T}^n)$ into $L_q(\mathbb{T}^n)$ if for an arbitrary function $f \in L_p(\mathbb{T}^n)$ with Fourier series $\sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{ikx}$ there exists a function $f_\lambda \in L_q(\mathbb{T}^n)$ with Fourier series $\sum_{k\in\mathbb{Z}^n} \lambda_k \hat{f}(k)e^{ikx}$ and if T_λ, $T_\lambda f = f_\lambda$, is a bounded operator from $L_p(\mathbb{T}^n)$ into $L_q(\mathbb{T}^n)$. The set $\mathbf{m}_p^q$ of these multipliers is a normed linear space with the norm

$$\|\lambda\|_{\mathbf{m}_p^q} = \|T_\lambda\|_{L_p\to L_q}.$$

In this paper, we find lower and upper bounds for the norm of multipliers $\lambda \in \mathbf{m}_p^q$. The methods used in our proofs are based on the results in [1, 2] establishing new inequalities describing the relationship between metric properties of a function and the summability of its Fourier coefficients with respect to the trigonometric system.

The following result by Hörmander is well known [3]: if $1 < p \leqslant 2 \leqslant q < \infty$ and $1/r = 1/p - 1/q$, then the embedding $l_{r\infty}(\mathbb{Z}^n) \hookrightarrow \mathbf{m}_p^q$ holds, that is,

$$\|\lambda\|_{\mathbf{m}_p^q} \leqslant c \sup_{e\in E} \frac{1}{|e|^{1-1/r}} \left| \sum_{m\in e} \lambda_m \right|.$$

Here $l_{r\infty}(\mathbb{Z}^n)$ is a discrete Lorentz space, E is the set of all finite subsets in $\mathbb{Z}^n$, and $|e|$ is the cardinality of a set e.

The assertion below is a refinement of Hörmander's theorem for multiple series ($n \geqslant 2$).

Theorem 1. *Suppose that* $1 < p \leqslant 2 \leqslant q < \infty$, $p' = p/(p-1)$, *and* $(j_1, \ldots, j_n)$ *is an arbitrary permutation of the sequence* $1, \ldots, n$. *Then the inequality*

$$\|\lambda\|_{\mathbf{m}_p^q} \leqslant C \sup_{k\in\mathbb{N}^n} \frac{1}{(\prod_{i=1}^n k_i)^{1/q+1/p'}} \sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \lambda_{m_1\ldots m_n}^{*j_1\ldots *j_n}$$

holds, where the sequence $\{\lambda_{m_1\ldots m_n}^{*j_1\ldots *j_n}\}_{m\in\mathbb{N}^n}$ *is obtained from* $\{|\lambda_r|\}_{r\in\mathbb{Z}^n}$ *by passing consecutively to the nondecreasing rearrangement with respect to the indices* $r_{j_1}, \ldots, r_{j_n}$ *and* C *is a constant depending only on the parameters* p *and* q *and the dimension* n.

Let $B = \{m_0, m_0+1, \ldots, m_0+l\}$ be an interval in $\mathbb{Z}$, and let $d \in \mathbb{N}$, $d > l$. A set of the form $I = \bigcup_{k=0}^N [B+kd] = \bigcup_{k=0}^N \{m+kd : m \in B\}$ will be called a *harmonic interval* in $\mathbb{Z}$. Accordingly, a set of the form $I = I_1 \times \cdots \times I_n$, where the I_i are harmonic intervals in $\mathbb{Z}$, will be called a harmonic interval in $\mathbb{Z}^n$.

Theorem 2. *Let* $1 < p \leqslant 2 \leqslant q < \infty$, *let* $p' = p/(p-1)$, *and let* M_0 *be the set of all harmonic intervals in* $\mathbb{Z}^n$. *If* $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}^n} \in \mathbf{m}_p^q$, *then*

$$\sup_{Q\in M_0} \frac{1}{|Q|^{1/q+1/p'}} \left| \sum_{m\in Q} \lambda_m \right| \leqslant C\|\lambda\|_{\mathbf{m}_p^q}.$$

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Definition. We say that a sequence $\{a_k\}_{k\in\mathbb{Z}^n}$ of complex numbers is *monotone in the extended sense* if there exists a number $c > 0$ such that

$$|a_k| \leqslant \frac{c}{|Q_k|}\left|\sum_{r\in Q_k} a_r\right|$$

for every $k \in \mathbb{Z}^n$, where $Q_k = \{r \in \mathbb{Z}^n : 0 \leqslant |r_j| \leqslant |k_j|,\ j = 1,\dots,n\}$.

If a sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{N}^n}$ is monotone or quasimonotone with respect to each k_j, then it is monotone in the extended sense.

Corollary 1. *Let $1 < p \leqslant 2 \leqslant q < \infty$ and $p' = p/(p-1)$. Suppose that $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}^n}$ is a monotone sequence in the extended sense. Then a necessary and sufficient condition for the relation $\lambda \in \mathbf{m}_p^q$ to hold is*

$$F(\lambda) = \sup_{m\in\mathbb{N}^n} \frac{1}{(m_1\cdots m_n)^{1/q+1/p'}}\left|\sum_{k_1=-m_1}^{m_1}\cdots\sum_{k_n=-m_n}^{m_n}\lambda_{k_1\dots k_n}\right| < \infty.$$

In this case, $\|\lambda\|_{\mathbf{m}_p^q} \sim F(\lambda)$.

Consider the Shapiro sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ (see [4]) defined by the recursion formulas $\varepsilon_1 = 1$, $\varepsilon_{2k} = \varepsilon_k$, $\varepsilon_{2k+1} = (-1)^k\varepsilon_k$, $k \in \mathbb{N}$. It consists of the numbers ±1 and satisfies the estimate

$$\left|\sum_{m=0}^{N-1}\varepsilon_m e^{imx}\right| \leqslant cN^{1/2}, \qquad x \in [0, 2\pi).$$

Let $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{N}^n}$, where $\varepsilon_k = \varepsilon_{k_1}\cdots\varepsilon_{k_n}$.

Theorem 3. *Suppose that either $1 < p \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant q < \infty$, $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{N}^n}$ is the Shapiro sequence, $(j_1,\dots,j_n)$ is an arbitrary permutation of the sequence $1,\dots,n$, and*

$$\frac{1}{r} = \begin{cases} 1/2 + 1/q & \text{for } 2 \leqslant p \leqslant q < \infty, \\ 1/p' + 1/2 & \text{for } 1 < p \leqslant q \leqslant 2. \end{cases}$$

Then

$$C^{-1}(r,n)\sup_{\substack{r\in\mathbb{Z}^n\\ m\in\mathbb{N}^n}}\frac{1}{(m_1\cdots m_n)^{1/r}}\left|\sum_{1\leqslant k\leqslant m}\varepsilon_k\lambda_{k+r}\right| \leqslant \|\lambda\|_{\mathbf{m}_p^q} \leqslant C(r,n)\sup_{m\in\mathbb{N}^n}\frac{1}{(m_1\cdots m_n)^{1/r}}\sum_{1\leqslant k\leqslant m}\lambda^{*j_1\dots *j_n}_{k_1\dots k_n}. \tag{1}$$

The upper and lower bounds in Theorem 3 do not depend on the parameter p for $2 \leqslant p \leqslant q < \infty$ and on the parameter q for $1 < p \leqslant q \leqslant 2$. This means that if these parameters are varied, then the class $\mathbf{m}_p^q$ varies with preservation of relation (1).

This assertion solves the problem of finding sufficient conditions for λ to belong to the space of multipliers $\mathbf{m}_p = \mathbf{m}_p^p$ essentially depending on the parameter p. In this connection, we note that, according to [5], the classical theorems on multipliers are somewhat unnatural, since the sufficient conditions in them usually do not depend on p.

Corollary 2. *Let $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{N}^n}$ be the Shapiro sequence. Suppose that either $1 < p \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant q < \infty$. If the sequence $\{\varepsilon_k\lambda_k\}_{k\in\mathbb{N}^n}$ is monotone in the extended sense, then the relation $\{\lambda_k\}_{k\in\mathbb{N}^n} \in \mathbf{m}_p^q$ is equivalent to the inequality*

$$\sup_{m\in\mathbb{N}^n}\frac{1}{(m_1\cdots m_n)^{1/r}}\sum_{k_1=1}^{m_1}\cdots\sum_{k_n=1}^{m_n}|\lambda_{k_1\dots k_n}| < \infty,$$

where the number $1/r$ is defined in Theorem 3.

There are as "many" sequences satisfying the assumptions of Corollary 2 as sequences monotone in the extended sense. For example, if $\lambda = \{\lambda_k\}$ is monotone in the extended sense, then $\lambda_\varepsilon = \{\varepsilon_k\lambda_k\}$ satisfies the assumptions of Corollary 2, since $\{\varepsilon_k^2\lambda_k\}_{k\in\mathbb{N}^n} = \{\lambda_k\}_{k\in\mathbb{N}^n}$.

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