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SHORT COMMUNICATIONS

On Inequalities for the Fourier Transform of Functions from Lorentz Spaces

A. N. Kopezhanova[1*], E. D. Nursultanov[1], and L.-E. Persson[2***]**

[1] *Eurasian National University, Astana, Kazakhstan*

[2] *Lulea University of Technology, Sweden*

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1. Let

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx, \qquad x \in \mathbb{R},$$

be the Fourier transform of a function $f \in L_1(\mathbb{R})$. Inequalities relating the integral properties of functions and their Fourier transforms are well known.

Suppose that $1 < p < 2$, $p' = p/(p-1)$, and $0 < q \le \infty$. Then the following inequalities hold:

$$\|\widehat{f}\|_{L_{p'}(\mathbb{R})} \le c_1 \|f\|_{L_p(\mathbb{R})}, \tag{1}$$

$$\|\widehat{f}\|_{L_{p',q}(\mathbb{R})} \le c_2 \|f\|_{L_{p,q}(\mathbb{R})}, \tag{2}$$

where $L_{p,q}(\mathbb{R})$ is the classical Lorentz space. These inequalities are called the Hausdorff–Young and Hardy–Littlewood–Stein inequalities, respectively, [1]. There are similar inequalities for Fourier transform on the torus $[0,1]$, i.e., for $\widehat{f} = \{\widehat{f}_n\}$, where $\widehat{f}_n$ are the Fourier coefficients with respect to some orthonormal system.

Suppose that ω is a nonnegative function on $[0,\infty]$. The generalized Lorentz space $\Lambda_q(\omega,\mathbb{R})$ is the set of all measurable functions f on $\mathbb{R}$ such that:

- if $0 < q < \infty$, then

$$\|f\|_{\Lambda_q(\omega,\mathbb{R})} = \left(\int_0^\infty (f^*(t)\omega(t))^q \,\frac{dt}{t}\right)^{1/q} < \infty,$$

- if $q = \infty$, then

$$\|f\|_{\Lambda_\infty(\omega,\mathbb{R})} = \sup_{t>0} f^*(t)\omega(t),$$

*E-mail: kopezhanova@yandex.ru
**E-mail: er-nurs@yandex.ru
***E-mail: lars-erik.persson@ltu.se

where f^* is a nonincreasing rearrangement of the function f.

In the present paper, we study upper and lower bounds for the norms of the Fourier transform in generalized Lorentz spaces. For the Fourier transform on $\mathbb{R}$ and on the torus $[0,1]$, we obtain inequalities reverse to (1) and (2) in a certain sense.

Suppose that $\delta > 0$ and $\omega(t)$ is a nonnegative function on $[0,\infty)$. The classes of function A_δ and B_δ are defined as follows:

$$A_\delta = \{\omega(t) : \omega(t)t^{-1/2-\delta} \text{ is an increasing function, } \omega(t)t^{-(1-\delta)} \text{ is a decreasing function}\},$$
$$B_\delta = \{\omega(t) : \omega(t)t^{-\delta} \text{ is an increasing function, } \omega(t)t^{-1+\delta} \text{ is a decreasing function}\}.$$

Then the classes A and B are defined as follows:

$$A = \bigcup_{\delta>0} A_\delta, \qquad B = \bigcup_{\delta>0} B_\delta.$$

Note that the class B contains the class A.

The following theorem is valid.

Theorem 1. *Suppose that $0 < q \le \infty$ and $\omega(t)$ belongs to the class A. Then the following inequalities hold*:

$$c_1\|\overline{f}\|_{\Lambda_q(\omega,\mathbb{R})} \le \|\widehat{f}\|_{\Lambda_q(\mu,\mathbb{R})} \le c_2\|f\|_{\Lambda_q(\omega,\mathbb{R})}, \tag{3}$$

where

$$\overline{f}(t) = \frac{1}{t}\left|\int_0^t f(s)\,ds\right|, \qquad \mu(t) = t\omega(1/t).$$

The inequalities in (3) and further are understood in the sense that if the right-hand side of the inequality is finite, then the left-hand side is also finite and the corresponding inequality holds.

In particular, when $w(t) = t^{1/p}$, $1 < p < 2$, then

$$c_1\|\overline{f}\|_{L_{p,q}} \le \|\widehat{f}\|_{L_{p'q}} \le c_2\|f\|_{L_{p,q}}. \tag{4}$$

Note that the left-hand inequality in (4) also follows from results given in [2], [3], where net spaces are used.

We say that a function f defined on $\mathbb{R}$ is *general monotone* if there exists a constant $M > 0$ for which the following inequality holds:

$$|f(x)| \le M\frac{1}{2x}\left|\int_{-x}^x f(t)\,dt\right|, \qquad x > 0.$$

Corollary 1. *Suppose that f is a general monotone function, $0 < q \le \infty$, and $\omega(t)$ belongs to the class A. Then, in order that $\widehat{f}$ belong to $\Lambda_q(\mu,\mathbb{R})$, it is necessary and sufficient that f belong to $\Lambda_q(\omega,\mathbb{R})$, and the following relation be valid*:

$$\|f\|_{\Lambda_q(\omega,\mathbb{R})} \asymp \|\widehat{f}\|_{\Lambda_q(\mu,\mathbb{R})}.$$

2. Suppose that f is a periodic function with period 1, is integrable on $[0,1]$, $\Phi = \{\varphi_k\}_{k=1}^\infty$ is an orthonormal system, and

$$a_k = a_k(f) = \int_0^1 f(x)\overline{\varphi_k(x)}\,d\mu, \qquad k \in \mathbb{N},$$

are the Fourier coefficients of the function f with respect to the system $\Phi = \{\varphi_n\}_{n=1}^\infty$.

An orthonormal system $\Phi = \{\phi_k(x)\}_{k=1}^\infty$ is said to be *regular* if there exists a constant B such that:

1) for any interval e from $[0,1]$ and $k \in \mathbf{N}$, the following inequality holds:

$$\left|\int_e \phi_k(x)\,dx\right| \le B\min\left(|e|, \frac{1}{k}\right),$$

2) for any interval w from $\mathbf{N}$ and $t \in [0,1]$,

$$\left(\sum_{k\in w}\phi_k(\,\cdot\,)\right)^*(t) \le B\min\left(|w|, \frac{1}{t}\right),$$

where $(\sum_{k\in w}\phi_k(\,\cdot\,))^*(t)$ is a nonincreasing rearrangement of the function $\sum_{k\in w}\phi_k(x)$.

Examples of regular systems are all trigonometric systems, the Walsh system, the Price system, etc.

Theorem 2. *Suppose that $\Phi = \{\varphi_n\}_{n=1}^\infty$ is a regular system and*

$$f \overset{\text{a.e.}}{=} \sum_{n=1}^\infty a_k\phi_n.$$

Suppose that $1 \le q \le \infty$ and $\omega(t)$ belongs to the class B. Then

$$\left(\int_0^1 (\overline{f}(t)\omega(t))^q\,\frac{dt}{t}\right)^{1/q} \le c_1\left(\sum_{k=1}^\infty (a_k^*\mu(k))^q\frac{1}{k}\right)^{1/q},$$

where

$$\overline{f}(t) = \frac{1}{t}\left|\int_0^t f(s)\,ds\right|, \qquad \mu(k) = k\omega\left(\frac{1}{k}\right),$$

and a_k^* is a nonincreasing rearrangement of the coefficients $\{a_k\}$.

Note that if, in Theorems 1 and 2, $\overline{f}(t)$ is replaced by

$$\frac{1}{t}\int_0^t |f(s)|\,ds$$

or $|f(t)|$ or $f^*(t)$, then the corresponding inequalities are no longer valid (relevant examples have been constructed).

We say that a sequence of complex numbers $\{a_k\}_{k\in\mathbb{N}}$ is *general monotone* if there exists a constant $M > 0$ such that

$$|a_n| \le M\frac{1}{n}\left|\sum_{k=1}^n a_k\right|, \qquad n \in \mathbb{N}.$$

For $\omega \in A$, the following inequality was obtained in [4]:

$$\left(\sum_{k=1}^\infty (a_k^*\mu(k))^q\frac{1}{k}\right)^{1/q} \le \|f\|_{\Lambda_q(\omega,[0,1])} := \left(\int_0^1 (f^*(t)\omega(t))^q\,\frac{dt}{t}\right)^{1/q}.$$

Using this inequality and Theorem 2,, we can prove the following statement.

Corollary 2. *Suppose that $1 \le q \le \infty$, $\Phi = \{\varphi_k\}_{k=1}^\infty$ is a regular system, $f \overset{\text{a.e.}}{=} \sum_{k=1}^\infty a_k\varphi_k$, and $\omega(t) \in A$. If f is a general monotone function or $\{a_k\}$ is a general monotone sequence, then the following relation holds*:

$$\|f\|_{\Lambda_q(\omega,[0,1])} \asymp \left(\sum_{k=1}^\infty (a_k^*\mu(k))^q\frac{1}{k}\right)^{1/q},$$

where $\mu(k) = k\omega(1/k)$.

The condition that the weight ω belong to the class A in the case $\omega(t) = t^{1/p}$ implies the condition $1 < p < 2$. Thus, it follows from Corollary 2 that, for $1 < p < 2$ provided that $\{a_k\}$ a general monotone sequence, the function $f = \sum a_k\varphi_k$ satisfies the relation

$$\|f\|_{L_p}^p \asymp \sum_{k=1}^{\infty} k^{p-2}|a_k|. \tag{5}$$

In the case $p > 2$, we have constructed an example of a trigonometric series with general monotone coefficients for which relation (5) does not hold. This indicates the importance of the condition $\omega \in A$ (the idea of the construction of the example belongs to M. I. Dyachenko).

Theorem 3. *Suppose that $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is a regular system, $1 \le q \le \infty$, and $f \overset{\text{a.e.}}{=} \sum_{k=1}^{\infty} a_k\varphi_k$. If $\omega(t)$ belongs to the class B, then*

$$c_1\biggl(\sum_{k=1}^{\infty}(\overline{a}_k\mu(k))^q\frac{1}{k}\biggr)^{1/q} \le c_2\|f\|_{\Lambda_q(\omega,[0,1])} \le c_2\biggl(\sum_{k=1}^{\infty}(|k\Delta a_k|\mu(k))^q\frac{1}{k}\biggr)^{1/q},$$

where

$$\overline{a}_k = \frac{1}{k}\biggl|\sum_{m=1}^{k} a_m(f)\biggr|, \qquad \Delta a_k = a_k - a_{k+1}, \qquad \mu(k) = k\omega\biggl(\frac{1}{k}\biggr).$$

Corollary 3. *Suppose that $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ is a regular system, $f \overset{a.e.}{=} \sum_{k=1}^{\infty} a_k\varphi_k$, and $\omega(t)$ belongs to the class B. If the sequence of complex numbers $\{a_k\}$ satisfies the condition*:

$$\text{there exist} \quad C > 0,\, d \ge 1 \quad \text{such that} \quad \sum_{k=n}^{\infty}|\Delta a_k| \le C\overline{a}_{dn}, \quad n \in \mathbb{N}, \tag{6}$$

then the following relation is valid:

$$\|f\|_{\Lambda_1(\omega,[0,1])} \asymp \sum_{k=1}^{\infty}|a_k|\omega\biggl(\frac{1}{k}\biggr).$$

Note that the class of sequences satisfying condition (6) is wider than the class $GM({}_*\beta)$, which was introduced in [5].

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