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SHORT COMMUNICATIONS

On Interpolation and Embedding Theorems for the Spaces $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$

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1. INTERPOLATION METHOD FOR ANISOTROPIC SPACES

Let A_1 be a Banach space, and let A_2 be a functional Banach lattice. We let $\mathbf{A} = (A_1, A_2)$ denote the space of A_1-valued measurable functions such that $\|f\|_{A_1} \in A_2$; this space is equipped with the norm $\|f\|_{\mathbf{A}} = \|\|f(x)\|_{A_1}\|_{A_2}$.

Let $\mathbf{A}_0 = (A_1^0, A_2^0)$ and $\mathbf{A}_1 = (A_1^1, A_2^1)$ be two anisotropic spaces, $E = \{\varepsilon = (\varepsilon_1, \varepsilon_2) : \varepsilon_i = 0$ or $\varepsilon_i = 1,\ i = 1, 2\}$. For an arbitrary $\varepsilon \in E$, we define the space $\mathbf{A}_\varepsilon = (A_1^{\varepsilon_1}, A_2^{\varepsilon_2})$ with the norm

$$\|a\|_{\mathbf{A}_\varepsilon} = \|\|a\|_{A_1^{\varepsilon_1}}\|_{A_2^{\varepsilon_2}}.$$

A pair of anisotropic spaces $\mathbf{A}_0 = (A_1^0, A_2^0)$ and $\mathbf{A}_1 = (A_1^1, A_2^1)$ is said to be *consistent* if there is a linear Hausdorff space containing the spaces $\mathbf{A}_\varepsilon$, $\varepsilon \in E$, as subspaces.

For an element $a \in \sum_{\varepsilon\in E} A_\varepsilon$, we define the functional

$$K(\mathbf{t}, a; \mathbf{A}) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon}.$$

Let $\mathbf{0} < \theta = (\theta_1, \theta_2) < \mathbf{1}$, and let $\mathbf{0} < \mathbf{r} = (r_1, r_2) \le \infty$. For an arbitrary permutation $\star = (j_1, j_2)$ of the set $\{1, 2\}$, we let $\mathbf{A}_{\theta\mathbf{r}\star} = (A_\varepsilon; \varepsilon \in E)_{(\theta_1,\theta_2),(r_1,r_2),\star}$ denote the linear subset $\sum_{\varepsilon\in E} \mathbf{A}_\varepsilon$ whose elements satisfy the condition

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}\star}} = \left(\int_0^\infty \left(\int_0^\infty (t_{j_1}^{-\theta_{j_1}} t_{j_2}^{-\theta_{j_2}} K(\mathbf{t}, a; \mathbf{A}))^{r_{j_1}} \frac{dt_{j_1}}{t_{j_1}}\right)^{r_{j_2}/r_{j_1}} \frac{dt_{j_2}}{t_{j_2}}\right)^{1/r_{j_2}} < \infty.$$

Lemma 1 ([1]). *Suppose that $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \le \infty$, $\star = (j_1, j_2)$ is a permutation of the set $\{1, 2\}$, and $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ and $\mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ are two consistent families of Banach spaces. If there are two vectors $\mathbf{M}_0 = (M_1^0, M_2^0)$ and $\mathbf{M}_1 = (M_1^1, M_2^1)$ with positive components such that there is a linear operator $T\colon A_\varepsilon \to B_\varepsilon$ with the norm estimate $C_\varepsilon \prod_{i=1}^2 M_i^{\varepsilon_i}$ for any $\varepsilon \in E$, then*

$$T\colon \mathbf{A}_{\theta\mathbf{r}\star} \to \mathbf{B}_{\theta\mathbf{r}\star}$$

with the norm

$$\|T\|_{\mathbf{A}_{\theta\mathbf{r}\star}\to\mathbf{B}_{\theta\mathbf{r}\star}} \le \max_{\varepsilon\in E} C_\varepsilon \prod_{i=1}^2 (M_i^0)^{1-\theta_i} (M_i^1)^{\theta_i}.$$

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2. THE SPACES $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$ AND THEIR INTERPOLATION

For $1 \le p, \tau \le \infty$ and $\Omega \subset \mathbb{R}^n$ the *Lorentz space* $L_{pq}(\Omega)$ is defined as the set of functions $f(x)$ measurable on Ω with the finite norm

$$\|f\|_{L_{p\tau}(\Omega)} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^\tau \dfrac{dt}{t}\right)^{1/\tau} & \text{for } 1 \le p, \tau < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) & \text{for } 1 \le p \le \infty,\ \tau = \infty, \end{cases}$$

where $f^*(t)$ is a nonincreasing permutation of the function $f(x)$.

Let $-\infty < \sigma < \infty$, $1 \le q, p, \tau \le \infty$, and let $\star = (j_1, j_2)$ be a permutation of the set $\{1, 2\}$. We define the space $\overset{\star}{\mathfrak{L}}{}^{\sigma q}_{p\tau}(\Omega)$ as the set of sequences $a = \{a_k(x)\}_{k=-\infty}^{\infty}$, where the $a_k(x)$ are functions measurable on Ω with the finite norm: for $\star = (1, 2)$,

$$\|a\|_{\overset{\star}{\mathfrak{L}}{}^{\sigma q}_{p\tau}(\Omega)} = \left(\sum_{k=-\infty}^{\infty} (2^{\sigma k} \|a_k\|_{L_{p\tau}})^q\right)^{1/q};$$

and for $\star = (2, 1)$,

$$\|a\|_{\overset{\star}{\mathfrak{L}}{}^{\sigma q}_{p\tau}(\Omega)} = \left\| \left(\sum_{k=-\infty}^{\infty} (2^{\sigma k} a_k^*(\,\cdot\,))^q\right)^{1/q} \right\|_{L_{p\tau}}.$$

Theorem 1. *Suppose that $-\infty < \sigma_0 \ne \sigma_1 < +\infty$, $1 \le q_0, q_1 \le \infty$, $1 \le p_0 \ne p_1 \le \infty$, $1 \le \tau_0, \tau_1 \le \infty$, and $\star_\varepsilon = (j_1^{(\varepsilon_1,\varepsilon_2)}, j_2^{(\varepsilon_1,\varepsilon_2)})$ are permutations of the set $\{1, 2\}$ for all $\varepsilon = (\varepsilon_1, \varepsilon_2) \in E$. Then, for $0 < \theta_1, \theta_2 < 1$, $1 \le r_1, r_2 \le \infty$, and a permutation $\star = (j_1, j_2)$ of the set $\{1, 2\}$, the relation*

$$(\overset{\star_\varepsilon}{\mathfrak{L}}{}^{\sigma_{\varepsilon_2} q_{\varepsilon_2}}_{p_{\varepsilon_1} \tau_{\varepsilon_1}}(\Omega); \varepsilon = (\varepsilon_1, \varepsilon_2) \in E)_{(\theta_1,\theta_2),(r_1,r_2),\star} = \overset{\star}{\mathfrak{L}}{}^{\sigma r_2}_{p r_1}(\Omega)$$

holds, where

$$\sigma = (1 - \theta_2)\sigma_0 + \theta_2 \sigma_1, \qquad \frac{1}{p} = \frac{1 - \theta_1}{p_0} + \frac{\theta_1}{p_1}.$$

For $-\infty < \sigma < \infty$, $1 < p < \infty$, $1 \le q, \tau \le \infty$, and a permutation $\star = (j_1, j_2)$ of the set $\{1, 2\}$, we define the space $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$ as the set of functions $f(x)$ with the finite norm

$$\|f\|_{\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)} = \|\{f * \varphi\}\|_{\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)},$$

where $\varphi(\xi) = \{\varphi_k(\xi)\}_{k=0}^{\infty}$ is a partition of unity on the domain Ω and

$$(f * \varphi)(x) = \int_\Omega f(x - y)\varphi(y)\, dy$$

is the convolution of functions f and φ.

The space $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$ with $\star = (1, 2)$ is a space of Besov space $B^\sigma_{pq(\tau)}(\mathbb{R}^n)$ type (see [2], [3]); for $\star = (2, 1)$, this space is somewhat similar in definition to the Lizorkin–Triebel space $F^\sigma_{pq(\tau)}(\mathbb{R}^n)$ (see [3]), but, in contrast to this space, the operation of taking the convolution of a nonincreasing permutation is applied not to the function

$$f_{\sigma q}(x) = (\sum_{k=0}^{\infty} (2^{\sigma k} |(\varphi_k * f)(x)|)^q)^{1/q}$$

but to each of the functions $(\varphi_k * f)(x)$, $k = 0, 1, 2, \dots$.

Theorem 2. *Suppose that $W^{\sigma_{\varepsilon_2}}_{p_{\varepsilon_1}}(\Omega)$, $\varepsilon=(\varepsilon_1,\varepsilon_2)\in E$, are Sobolev spaces, where $\sigma_0\neq\sigma_1\in\mathbb{N}$, and $1\le p_0\neq p_1\le\infty$. Then, for $0<\theta_1,\theta_2<1$, $1\le r_1,r_2\le\infty$, and a permutation $\star=(j_1,j_2)$ of the set $\{1,2\}$, the relation*

$$(W^{\sigma_{\varepsilon_2}}_{p_{\varepsilon_1}}(\Omega);\varepsilon=(\varepsilon_1,\varepsilon_2)\in E)_{(\theta_1,\theta_2),(r_1,r_2),\star}=\overset{\star}{\mathfrak{B}}{}^{\sigma r_2}_{pr_1}(\Omega),$$

holds, where

$$\sigma=(1-\theta_2)\sigma_0+\theta_2\sigma_1,\qquad \frac{1}{p}=\frac{1-\theta_1}{p_0}+\frac{\theta_1}{p_1}.$$

Theorem 2 shows that the interpolation of Sobolev spaces leads to Besov- and Lizorkin–Triebel-type spaces depending on the parameter $\star$. This fact allows us to consider these spaces as a unified scale of spaces.

Theorem 3. *Suppose that $-\infty<\sigma_0\neq\sigma_1<+\infty$, $1\le q_0,q_1\le\infty$, $1\le p_0\neq p_1\le\infty$, $1\le\tau_0,\tau_1\le\infty$, and $\star_\varepsilon=(j_1^{(\varepsilon_1,\varepsilon_2)},j_2^{(\varepsilon_1,\varepsilon_2)})$ are permutations of the set $\{1,2\}$ for all $\varepsilon=(\varepsilon_1,\varepsilon_2)\in E$. Then, for $0<\theta_1,\theta_2<1$, $1\le r_1,r_2\le\infty$, and a permutation $\star=(j_1,j_2)$ of the set $\{1,2\}$, the relation*

$$(\overset{\star_\varepsilon}{\mathfrak{B}}{}^{\sigma_{\varepsilon_2}q_{\varepsilon_2}}_{p_{\varepsilon_1}\tau_{\varepsilon_1}}(\Omega);\varepsilon=(\varepsilon_1,\varepsilon_2)\in E)_{(\theta_1,\theta_2),(r_1,r_2),\star}=\overset{\star}{\mathfrak{B}}{}^{\sigma r_2}_{pr_1}(\Omega),$$

holds, where

$$\sigma=(1-\theta_2)\sigma_0+\theta_2\sigma_1,\qquad \frac{1}{p}=\frac{1-\theta_1}{p_0}+\frac{\theta_1}{p_1}.$$

Theorem 3 shows that the scale of spaces $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$ is closed with respect of a given interpolation method, i.e., the result of interpolation of spaces is described by a space from the same scale. Here we note that the scales of Besov and Lizorkin–Triebel spaces are not closed with respect to a real interpolation method.

3. EMBEDDING THEOREMS FOR DIFFERENT METRICS AND DIMENSIONS FOR THE SPACES $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\mathbb{R}^n)$

Applying Theorems 2 and 3 to the embedding theorems for different metrics and dimensions for Sobolev, Besov, or Lizorkin–Triebel spaces [2]–[4], we obtain the corresponding theorems for the spaces $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$.

Theorem 4. *Suppose that $-\infty<\sigma_0<\sigma_1<\infty$, $1<p_0<p_1<\infty$, $1\le q,\tau\le\infty$, and $\star=(j_1,j_2)$ is a permutation of the set $\{1,2\}$. Then the embedding*

$$\overset{\star}{\mathfrak{B}}{}^{\sigma_1 q}_{p_1\tau}(\mathbb{R}^n)\hookrightarrow\overset{\star}{\mathfrak{B}}{}^{\sigma_1 q}_{p_0\tau}(\mathbb{R}^n)$$

holds for

$$\sigma_0-\frac{n}{p_0}\le\sigma_1-\frac{n}{p_1}.$$

Let $x=(x_1,\dots,x_n)$; we set $x'=(x_1,\dots,x_{n-1})$. For functions f from the spaces $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\mathbb{R}^n)$, we consider the operator $\mathfrak{R}$ acting by the rule

$$\mathfrak{R}f(x')=\biggl\{f(x',0),\frac{\partial f}{\partial x_n}(x',0),\dots,\frac{\partial^r f}{\partial x_n^r}(x',0)\biggr\}.$$

Theorem 5. *Suppose that $1<p<\infty$, $1\le q,\tau\le\infty$, and $\star=(j_1,j_2)$ is a permutation of the set $\{1,2\}$; then, for $\sigma>r+1/p$, the operator $\mathfrak{R}$ is a retraction of $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\mathbb{R}^n)$ onto*

$$\prod_{j=0}^{r}\overset{\star}{\mathfrak{B}}{}^{(\sigma-1/p-j)q}_{p\tau}(\mathbb{R}^{n-1}).$$

We note that, in contrast to the Lizorkin–Triebel spaces, the scale of spaces $\overset{\star}{\mathfrak{B}}{}^{\sigma q}_{p\tau}(\Omega)$ introduced above is closed with respect to embedding theorems for different dimensions.

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