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CONDITIONS FOR EXISTENCE OF A GLOBAL STRONG SOLUTION TO ONE CLASS OF NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACE. II

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Abstract: We obtain a criterion of global strong solvability for one class of nonlinear evolution equations in Hilbert space.

Keywords: Navier–Stokes equation, global strong solution

In this continuation of [1] we apply the results of [1] to the system of Navier–Stokes equations.

1. Some available facts about classes of periodic functions on a cube. Take a cube Q with side length 2π, the Laplace operator Δ, some $p \in (1; \infty)$, and some $\alpha \in (-\infty; +\infty)$. Define the norm

$$|u|_{W^{\alpha}_{p,\pi}(Q)} = |(-\Delta + E)^{\frac{\alpha}{2}} u|_{L_p(Q)} \tag{1}$$

on the set of trigonometric polynomials. Here E is the identity operator.

Denote by $W^{\alpha}_{p,\pi}(Q)$ the norm completion of the linear space of trigonometric polynomials. This is a fractional Sobolev space of periodic functions.

If in (1) we consider $\mathbb{R}^3$ instead of the cube Q then $W^{\alpha}_{p}(\mathbb{R}^3)$ is the space of Bessel potentials as discussed in [2]. The norm on $W^{\alpha}_{p}(\mathbb{R}^3)$ is defined as

$$|u|_{W^{\alpha}_{p}(\mathbb{R}^3)} = \left[\int\limits_{\mathbb{R}^3} \left|F^{-1}_{\xi\to y}(|\xi|^2+1)^{\frac{\alpha}{2}} F_{x\to\xi} u(x)\right|^p dy\right]^{\frac{1}{p}}, \tag{2}$$

where F is the Fourier transform, and F^{-1} is the inverse Fourier transform.

Theorem 1. *Take* $1 < p \le q < \infty$ *and* $\alpha \ge \beta \ge 0$ *such that* $\alpha - \beta \ge 3\big(\frac{1}{p} - \frac{1}{q}\big)$. *Then*

$$W^{\alpha}_{p}(\mathbb{R}^3) \hookrightarrow W^{\beta}_{q}(\mathbb{R}^3),$$

where the symbol $\hookrightarrow$ *means a topological embedding.*

The proof of this theorem has a long history. A complete proof (in a more general form) is due to Lizorkin [3].

For the sake of convenience we state one result that follows from Theorem 1.

Lemma 1. *Take* $1 < p \le q < \infty$ *and* $\gamma = 3\big(\frac{1}{p} - \frac{1}{q}\big)$. *Then*

$$|u|_{L_q(\mathbb{R}^3)} \le C|(-\Delta + E)^{\frac{\gamma}{2}} u|_{L_p(\mathbb{R}^3)}, \quad |(-\Delta + E)^{-\frac{\gamma}{2}} u|_{L_q(\mathbb{R}^3)} \le C|u|_{L_p(\mathbb{R}^3)}.$$

For the fractional Sobolev classes $W^{l}_{p,\pi}$ of periodic functions we have the propositions similar to Theorem 1 and Lemma 1.

Lemma 1′. (a) *Take* $1 < p \le 2$ *and* $\gamma = 3\big(\frac{1}{p} - \frac{1}{2}\big)$. *Then* $|(-\Delta + E)^{-\frac{\gamma}{2}} u|_{L_2(Q)} \le C|u|_{L_p(Q)}$.

(b) *Take* $2 \le q < \infty$ *and* $\gamma = 3\big(\frac{1}{2} - \frac{1}{q}\big)$. *Then* $|u|_{L_q(Q)} \le C|(-\Delta + E)^{\frac{\gamma}{2}} u|_{L_2(Q)}$.

The books on embedding theorems pay much attention to the theory of $W^{\alpha}_{p}(\mathbb{R}^3)$; Theorem 1 is stated and proved there. However, an analog of Theorem 1 for the spaces of functions periodic on Q is absent

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from practically all available books. Certainly, the conclusions of this lemma can be extracted from the result of Lizorkin [3].

Let us point out another approach. In the $L_p(Q)$ spaces for $1 < p < \infty$ the operators $(-\Delta + E)^\alpha$ with $\alpha > 0$ can be defined (see [4, p. 308]) as

$$(-\Delta + E)^\alpha u = \frac{\sin \alpha\pi}{\pi} \int\limits_0^\infty [\lambda^{\alpha-1}(\lambda E - \Delta + E)^{-1}(-\Delta + E)u]\, d\lambda, \tag{3}$$

where $u \in D(-\Delta)$ and $D(\cdot)$ means the domain of definition. Here Δ is the selfadjoint extension of the Laplace operator corresponding to periodic boundary conditions. The equality (3), initially defined on trigonometric polynomials, extends by continuity in the metric of $L_p(Q)$ to all $u(\cdot)$ for which the right-hand side of (3) makes sense. Since the local principal part of the integral operator $(\lambda E - \Delta + E)^{-1}$ or the Green's function is the same for $\mathbb{R}^3$ and Q and equals a fundamental solution; Lemma $1'$ reduces to Lemma 1.

Lemma $1'$ can be proved without using Theorem 1 or Lemma 1. For instance, Theorem 2 of Nursultanov in [5] enables us to prove an analog of Theorem 1 for periodic classes by the Abel transform and the Young–O'Neil inequalities.

Lemma 2. *Take some trigonometric polynomials u and v, some $-\infty < \gamma < \frac{3}{4}$, and some first order differential operator D with constant coefficients. Then*

$$|(-\Delta + E)^{-\gamma}(u \cdot Dv)|_{L_2(Q)} \le C|(-\Delta + E)^{\frac{3}{8}-\frac{\gamma}{2}}u|_{L_2(Q)} \cdot |(-\Delta + E)^{\frac{7}{8}-\frac{\gamma}{2}}v|_{L_2(Q)}.$$

Proof. For $\frac{3}{2} \ge 2\gamma = 3\left(\frac{1}{p} - \frac{1}{2}\right) \ge 0$ in Lemma $1'$ we have

$$|(-\Delta + E)^{-\gamma}(u \cdot Dv)|_{L_2(Q)} \le C|u \cdot Dv|_{L_p(Q)}.$$

Use the Cauchy inequality and Lemma $1'$ to obtain

$$\begin{aligned} |(-\Delta + E)^{-\gamma}(u \cdot Dv)|_{L_2(Q)} &\le C|u|_{L_{2p}(Q)}|Dv|_{L_{2p}(Q)} \\ &\le C|(-\Delta + E)^{\frac{3}{8}-\frac{\gamma}{2}}u|_{L_2(Q)}|(-\Delta + E)^{\frac{7}{8}-\frac{\gamma}{2}}v|_{L_2(Q)}. \end{aligned}$$

This proves the lemma for $\gamma \ge 0$.

For $\gamma \in \left[-\frac{1}{2}, 0\right)$ we use the already proven part of the lemma and

$$\begin{aligned} &|(-\Delta + E)^{-\gamma} f|_{L_2(Q)} = |[(-\Delta + E)^{-(\gamma+\frac{1}{2})}](-\Delta + E)^{\frac{1}{2}} f|_{L_2(Q)} \\ &\le c\sum_{j=1}^{3}\left|[(-\Delta + E)^{-(\gamma+\frac{1}{2})}]\frac{\partial}{\partial x_j} f\right|_{L_2(Q)} + |[(-\Delta + E)^{-(\gamma+\frac{1}{2})}] f|_{L_2(Q)}, \end{aligned}$$

$$\sum_{j=1}^{3}\left|[(-\Delta + E)^{-(\gamma+\frac{1}{2})}]\frac{\partial}{\partial x_j} f\right|_{L_2(Q)} + |[(-\Delta + E)^{-(\gamma+\frac{1}{2})}] f|_{L_2(Q)} \le c|(-\Delta + E)^{-\gamma} f|_{L_2(Q)}.$$

These inequalities follow from simple relations between the Fourier coefficients.

Denote by $L_{p,\alpha}$ the space of vector functions $u(x) = (u_1(x), u_2(x), u_3(x))$ of $x \in Q$ whose all components belong to $W^\alpha_{p,\pi}(Q)$. These spaces, as well as $W^\alpha_{p,\pi}(Q)$, are defined for all $p \in (1, \infty)$ and $\alpha \in (-\infty, \infty)$.

Put $(u, \nabla)v = u_i\partial_i v = (u_i\partial_i v_1, u_i\partial_i v_2, u_i\partial_i v_3)$, where $\partial_i = \frac{\partial}{\partial x_i}$ and $u_i\partial_i = u_1\partial_1 + u_2\partial_2 + u_3\partial_3$.

From Lemma 2 we immediately deduce

Lemma 3. *Take some three-component vector functions u and v and $-\infty < \gamma < \frac{3}{4}$. Then*

$$|(-\Delta + E)^{-\gamma}(u,\nabla)v|_{L_{2,0}} \le C|(-\Delta + E)^{\frac{3}{8}-\frac{\gamma}{2}}u|_{L_{2,0}}|(-\Delta + E)^{\frac{7}{8}-\frac{\gamma}{2}}v|_{L_{2,0}}.$$

A basis of the $L_{p,\alpha}$ space consists of the elements

$$(\,1\ 0\ 0\,)\,e^{i\langle m,x\rangle}, (\,0\ 1\ 0\,)\,e^{i\langle m,x\rangle}, (\,0\ 0\ 1\,)\,e^{i\langle m,x\rangle}, \tag{4}$$

where m runs over the set of all integer vectors, and $\langle m,x\rangle = m_1x_1 + m_2x_2 + m_3x_3$, with $i^2 = -1$. From a linear combination of (4) for $m \ne (0,0,0)$ we arrange two linearly independent pairwise orthogonal vectors

$$e_m^{(j)} = \left(\,c_{m_1}^{(j)}\ c_{m_2}^{(j)}\ c_{m_3}^{(j)}\,\right) e^{i\langle m,x\rangle} \quad (j = 1,2)$$

that satisfy the conditions

$$c_{m_1}^{(j)}m_1 + c_{m_2}^{(j)}m_2 + c_{m_3}^{(j)}m_3 = 0, \quad \sum_{k=1}^{3}\left|c_{m_k}^{j}\right|^2 = \left(\frac{1}{2\pi}\right)^3. \tag{4$'$}$$

From the set of vector functions $e_m^{(1)}$ and $e_m^{(2)}$, where $m = (m_1, m_2, m_3)$ runs over the set of all integer vectors ($m \ne (0,0,0)$, and the vectors (4) for $m = (0,0,0)$ we make the orthonormal system

$$(\,1\ 0\ 0\,), (\,0\ 1\ 0\,), (\,0\ 0\ 1\,), \dots, e_m^{(1)}, e_m^{(2)}, \dots \tag{4$''$}$$

which is a subsystem of (4).

For $p \in (1,\infty)$ and $\alpha \in (-\infty,+\infty)$ we introduce the space $H_{p,\alpha} \equiv H_{p,\alpha}(Q)$ as the $L_{p,\alpha}$-completion of all possible finite linear combinations of the elements of (4$''$). This yields a Banach space, which is a subspace of $L_{p,\alpha}$.

Lemma 4. (a) *Take some finite linear combination $u(x) = (u_1(x), u_2(x), u_3(x))$ of the elements of* (4$''$). *Then*

$$\operatorname{div} u(x) \equiv \sum_{j=1}^{3} \frac{\partial u_j(x)}{\partial x_j} = 0. \tag{5}$$

(b) *Take some finite linear combination $u(x)$ of the elements of* (4) *and suppose that* (5) *holds. Then $u(x)$ can be represented as a finite sum of elements of the form* (4$''$).

Because of (4$'$) this lemma follows easily from the construction of (4$''$). Using the standard techniques, from Lemma 4 we deduce

Lemma 5. $H_{p,\alpha} \equiv H_{p,\alpha}(Q) = \{u \in L_{p,\alpha}(Q) : \operatorname{div} u = 0\}$.

Here $\operatorname{div} u = 0$ is understood in the usual sense for $\alpha \ge \frac{1}{2}$, and for $\alpha < \frac{1}{2}$ it is understood to be the vanishing of a generalized vector function over finite linear combinations of the basis elements (4).

Lemma 6. *Take some $\alpha \in (-\infty,+\infty)$ and $u \in L_{2,\alpha}$. Then $(u + \operatorname{grad}(-\widetilde{\Delta})^{-1}\operatorname{div} u) \in H_{2,\alpha}$.*

Here $(-\widetilde{\Delta})^{-1}$ is the operator defined as follows: If

$$u = \sum_{\{m\}} (\,c_{m,1}\ c_{m,2}\ c_{m,3}\,)\,e^{i\langle m,x\rangle}$$

then

$$(-\widetilde{\Delta})^{-1}u = \sum_{\{m\},|m|\ne 0} \frac{1}{|m|^2}(\,c_{m,1}\ c_{m,2}\ c_{m,3}\,)\,e^{i\langle m,x\rangle}.$$

The proof of the lemma consists in simple straightforward calculations. For $\alpha < \frac{1}{2}$ the operations are understood in the sense of distributions.

Lemma 7. *Take some $P(x) \in W^{\alpha}_{2,\pi}(Q)$ and $\alpha \in (-\infty, +\infty)$. Then*

$$\operatorname{grad} P(x) + [\operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div}] \operatorname{grad} P(x) = 0.$$

The proof consists in simple calculations.

It is easy to prove

Lemma 8. *$L_{2,\alpha}(Q) = H_{2,\alpha}(Q) \oplus M$, where $\oplus$ means the orthogonal sum, and $M = \{u : u = \operatorname{grad} g \in L_{2,\alpha}(Q)\}$, i.e., if $f(x) \in L_{2,\alpha}(Q)$ then $f(x) = \tilde{f}(x) + u(x)$ for $\tilde{f}(x) \in H_{2,\alpha}(Q)$ and $u = \operatorname{grad} g \in M$.*

The proof follows from the equality

$$f(x) = [f + \operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div} f] - \operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div} f$$

and Lemma 6.

The following remark is just curious and possibly useful in hydrodynamics.

Remark 1. Take two domains Ω_0 and Ω_1 in $\mathbb{R}^3$ such that Ω_1 includes Ω_0 together with some neighborhood. Consider in Ω_1 some boundary value problem

$$\Delta u(x) = f(x),\ x \in \Omega_1, \quad R(u)|_{\partial\Omega_1} = 0, \tag{6}$$

where $R(\cdot)$ is a linear operator. Suppose that this problem is uniquely solvable for all $f \in L_2(\Omega_1)$ and its solution belongs to $W_2^{(2)}(\Omega_1)$. Denote by $\widehat{\Delta}^{-1}$ the operator that associates to f the solution to (6). On the space of three-component vector functions with components in $W_2^1(\Omega_0)$ define the operator

$$Nu \equiv \operatorname{grad}(\widehat{\Delta}^{-1})\chi(\Omega_0) \operatorname{div} u,$$

where χ is the characteristic function. It is easy to verify that $\operatorname{div}(E - N)u \equiv \operatorname{div} u - \operatorname{div} Nu = 0$ in Ω_0.

2. On the system of Navier–Stokes equations. Consider the system of Navier–Stokes equations

$$\frac{\partial}{\partial t}u_j - \Delta u_j + \sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k} u_j + \frac{\partial}{\partial x_j} P = f_j,\ j = 1,2,3, \quad \operatorname{div} u \equiv \sum_{k=1}^{3} \frac{\partial}{\partial x_k} u_k = 0, \tag{7}$$

in $[0,a] \times Q$, where $t \in [0,a]$ and $x \in Q$. Add to the system some boundary conditions periodic with respect to the spatial variables and the zero Cauchy initial condition with respect to time:

$$\begin{gathered} u_j(t,x)|_{t=0} = 0, \quad u_j(t,x)|_{x_k=-\pi} = u_j(t,x)|_{x_k=\pi}, \\ \frac{\partial}{\partial x_k} u_j(t,x)\Big|_{x_k=-\pi} = \frac{\partial}{\partial x_k} u_j(t,x)\Big|_{x_k=\pi}, \\ P(t,x)|_{x_k=-\pi} = P(t,x)|_{x_k=\pi} \quad (k,j = 1,2,3). \end{gathered} \tag{8}$$

From (7) and (8) we cannot determine the pressure $P(t,x)$ uniquely: it may involve as a summand an arbitrary function of time t. Therefore, in order to determine P uniquely, add to (7) and (8) the condition

$$\int_Q P(t,x)\, dx = 0. \tag{9}$$

Act on the first three equations in (7) by $K \equiv E + \operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div}$. This yields

$$\frac{\partial u}{\partial t} - \Delta u + (E + \operatorname{grad}(-\tilde{\Delta})^{-1} \operatorname{div})(u, \nabla)u = (E + \operatorname{grad}(-\tilde{\Delta})^{-1} \operatorname{div})f \equiv \tilde{f}, \quad u|_{t=0} = 0. \tag{10}$$

In deriving (10) we used (7), (8), and the fact that $\operatorname{div} u = 0$ implies

$$(E + \operatorname{grad}(-\widetilde{\Delta})^{-1}\operatorname{div})\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}, \quad (E + \operatorname{grad}(-\widetilde{\Delta})^{-1}\operatorname{div})\Delta u = \Delta u.$$

Since on the class of periodic functions (the completion of the space of trigonometric polynomials) the operators $\frac{\partial}{\partial t}$ and Δ commute with the projection $K = [E + \operatorname{grad}(-\widetilde{\Delta})^{-1}\operatorname{div}]$, in the system of equations (10) we need not write Ku, $K\frac{\partial}{\partial t}K$, and $K\Delta K$ instead of u, $\frac{\partial}{\partial t}$, and Δ respectively.

Note that, by acting on (10) with the divergence operator, we obtain

$$\frac{\partial}{\partial t}(\operatorname{div} u) - \Delta(\operatorname{div} u) = 0, \quad (\operatorname{div} u)|_{t=0} = 0.$$

This implies that every solution to (10) satisfies the condition $\operatorname{div} u = 0$ automatically.

As the spaces H from Sections 1–3 in [1] we take $H_{2,0}$. Then $(-\Delta)$ is a nonnegative selfadjoint operator on H, and $[E + \operatorname{grad}(-\tilde{\Delta})^{-1}\operatorname{div}](u,\nabla)u$ is a bilinear operator on H. Denote them by A and $B(\cdot,\cdot)$.

We can now rewrite (10) as

$$u'_t + Au + B(u,u) = f(t), \quad u|_{t=0} = 0. \tag{10$'$}$$

The conditions (2) and (3) in [1] are satisfied because of Lemma 3. Aside from the conditions of Sections 1–3 in [1], another condition is satisfied:

$$\langle B(u,g), g\rangle_{H_{2,0}} = 0.$$

This condition on the solution to (10′) implies the well-known a priori estimate

$$|u|^2(t) + \int_0^t |A^{\frac{1}{2}}u|^2\,dt \le \int_0^t |f|^2\,dt. \tag{11}$$

This estimate enables us to deduce the existence of a global weak Hopf solution; see [6, p. 83].

Apply Theorem A in [1] now to the system of equations (10). Pick $a > 0$. Denote by $H^a_{2,\alpha}$ the space of three-component vector functions $u(t,x)$ whose values belong to $H_{2,\alpha}$ for almost all $t \in [0,a]$, and such that

$$|u(\cdot,\cdot)|_{H^a_{2,\alpha}} = \left(\int_0^a |u(t,\cdot)|^2_{H_{2,\alpha}}\,dt\right)^{\frac{1}{2}} < \infty.$$

Consider in $H^a_{2,0}$ the system

$$\begin{gathered}\frac{\partial v}{\partial t} - \Delta v + K(v,\nabla)v = s\ (0 < t < a), \quad v|_{t=0} = 0,\\ -\frac{\partial s}{\partial t} - \Delta s - K\left[(v,\nabla)s + \sum_{i=1}^3 s_i \operatorname{grad} v_i\right] = 0\ (0 < t < a), \quad s|_{t=a} = 0.\end{gathered} \tag{12}$$

Here $K = [E + \operatorname{grad}(-\widetilde{\Delta})^{-1}\operatorname{div}]$ is a projection.

Remark 2. Note that in (12) the solutions are generalized; their meaning is given in Definition 1; see below. Therefore, the time derivatives of $s(\cdot)$ are generalized derivatives; the test functions are defined by (13). We can show that solutions to (12) on every compactly supported subinterval of $[0,a]$ have the ordinary derivatives and infinite smoothness.

Let us give the definition of a generalized solution. Note first that the a priori estimate (11) implies (see [1]) that if $(w_1,w_2) \in D_a$ and $w' + Aw + B(w,w) \in H_2[0,a]$ with $w(0) = 0$ then $B_w w_1 \in H_2[0,a]$.

Therefore, in Definition 3d in [1] instead of saying "if $(w_1, w_2) \in D_a$ and $B_w w_1 \in H_2[0,a]$ then ..." we can just say "if $(w_1, w_2) \in D_a$ then ..." Taking this into account, we arrive at the following definition of a generalized solution to (12).

DEFINITION 1. Take

$$w_1(t,x) = \sum c_{1,n}(t) e_n(x), \quad w_2(t,x) = \sum c_{2,n}(t) e_n(x), \tag{13}$$

where the sums are finite, $c_{1,n}(t)$ and $c_{2,n}(t)$ are infinitely smooth functions on $[0,a]$ that satisfy the conditions $c_{1,n}(0) = 0$ and $c_{2,n}(a) = 0$, and $\{e_n(x)\}$ is the orthonormal basis $(4'')$. A pair

$$v(t,x) = \sum_{n=1}^{\infty} v_n(t) e_n(x), \quad s(t,x) = \sum_{n=1}^{\infty} s_n(t) e_n(x) \tag{14}$$

of functions is called a *solution to* (12) if

$$\int_0^a \int_Q (|s(t,x)|^2 + |v(t,x)|^2)\, dxdt < \infty,$$

for every $\delta \in (0,a)$ the inequality

$$\int_0^{a-\delta} \int_Q |v'_t(t,x) - \Delta v(t,x)|^2\, dxdt < \infty,$$

is satisfied, and also for every pair (w_1, w_2) of the form (13) we have

$$\int_0^a \int_Q s(t,x)[w'_{1t} - \Delta w_1 + (v,\nabla)w_1 + (w_1,\nabla)v]\, dxdt = 0,$$

$$\int_0^a \int_Q \{[-w'_{2t} - \Delta w_2] v(t,x) + w_2 (v,\nabla) v(t,x) - w_2(t,x) s(t,x)\}\, dxdt = 0. \tag{15}$$

In the last equalities in front of $(v,\nabla)w_1 + (w_1,\nabla)v$ and $(v,\nabla)v(t,x)$ we must insert the projection $K = [E + \operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div}]$. However, this operator "is switched" to the other factor on which it acts as the identity. Therefore, the projection K is absent from these equalities.

Theorem 2. *For every $\tilde{f} \in H^a_{2,0}$ the system* (10) *has a solution $u(t,x)$ that satisfies the conditions $\frac{\partial u}{\partial t}$, $\Delta u \in H^a_{2,0}$ if and only if* (12) *has only the trivial solution in $H^a_{2,0}$; i.e., $v(t,x) \equiv s(t,x) \equiv 0$.*

PROOF. By Definition 1, (12) is equivalent to the system (4) from Theorem A in [1] that corresponds to $(10')$; i.e., (4) in [1] is an abstract form of (12). Therefore, Theorem 2 is a corollary to Theorem A in [1].

The following theorem is a rephrasal of Theorem 2.

Theorem 2′. *Take some $f = (f_1(t,x), f_2(t,x), f_3(t,x))$ satisfying*

$$\int_0^a |f(t,\cdot)|^2_{L_2(Q)}\, dt < \infty.$$

In order for (7)–(9) *to have a solution (u, P), where $u = (u_1(t,x), u_2(t,x), u_3(t,x))$, that satisfies the condition*

$$\int_0^a |u_t - \Delta u|^2_{L_2(Q)} + \int_0^a |\operatorname{grad} P|^2_{L_2(Q)}\, dt < \infty$$

it is necessary and sufficient that the generalized solution can only be trivial to the system of equations

$$
\begin{gathered}
\frac{\partial v_j}{\partial t} - \Delta v_j + u_k \frac{\partial}{\partial x_k} v_j + \frac{\partial}{\partial x_j} P = s_j \ (j = 1, 2, 3), \quad \operatorname{div} v = \frac{\partial}{\partial x_k} v_k = 0, \\
-\frac{\partial s_j}{\partial t} - \Delta s_j - \left(\frac{\partial}{\partial x_j} s_k \right) v_k - \frac{\partial}{\partial x_k} s_j v_k + \frac{\partial}{\partial x_j} r = 0, \\
\operatorname{div} s = \frac{\partial}{\partial x_k} s_k = 0 \ (j = 1, 2, 3), \quad \int\limits_Q P(t, x) = 0, \ \int\limits_Q r(t, x) = 0
\end{gathered} \tag{16}
$$

with the Cauchy condition

$$
v(t, x)|_{t=0} = 0, \quad s(t, x)|_{t=a} = 0
$$

and with periodic boundary conditions on $v = (v_1, v_2, v_3)$, $s = (s_1, s_2, s_3)$, P, r, $\frac{\partial}{\partial x_k} v$, *and* $\frac{\partial}{\partial x_k} s$, *where* $k = 1, 2, 3$, *with respect to the spatial variables.*

Proof of Theorem $2'$. Take some solution $(u(t, x), P(t, x))$ to (7)–(9). Then $u(t, x)$ is a solution to (10) because by acting on (7), while taking into account (8) and (9), with the projection

$$
K = [E + \operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div}],
$$

we obtain (10). Conversely, if $u(t, x)$ is a solution to (10) written as in (14); then, putting

$$
P(t, x) = c(t) + (-\widetilde{\Delta})^{-1} \operatorname{div}(u, \nabla) u - (-\widetilde{\Delta})^{-1} \operatorname{div} f, \tag{17}
$$

where $c(t)$ is an arbitrary function of time, instead of (10) we obtain

$$
\frac{\partial u}{\partial t} - \Delta u + (u, \nabla) u + \operatorname{grad} P = f, \quad u|_{t=0} = 0.
$$

In (17) the operator $(-\widetilde{\Delta})^{-1} \operatorname{div}$ is understood to be the closure of $(-\widetilde{\Delta})^{-1} \operatorname{div}$ defined initially on trigonometric polynomials.

The equality $\operatorname{div} u = 0$, as well as periodic boundary conditions on u and P with respect to the spatial variables, follows from the representability of u in the form (14) and the definition of $(-\widetilde{\Delta})^{-1} \operatorname{div}$. Consequently, (7)–(9) is equivalent to (10) in the class of vector functions representable as in (14).

Acting on (16) with the projection $E + \operatorname{grad}(-\widetilde{\Delta})^{-1} \operatorname{div}$ and taking into account that $\operatorname{div} s = \operatorname{div} v = 0$, as well as the periodic boundary conditions with respect to the spatial variables, we get (12); for v and s we obtain representations of the form (14). Thus, (16) implies (12).

Conversely, (12) implies (16). Indeed, writing

$$
P(t, x) = c_1(t) + (-\widetilde{\Delta})^{-1} \operatorname{div}(v, \nabla) v,
$$

$$
r(t, x) = c_2(t) + (-\widetilde{\Delta})^{-1} \operatorname{div} \left[(v, \nabla) s + \sum_{i=1}^{3} s_i \operatorname{grad} v_i \right],
$$

we derive (16) from (12). The equation $\operatorname{div} s = \operatorname{div} v = 0$, as well as the periodicity of s, v, and r with respect to the spatial variables, follows from the representability of s and v in the form (14). We have proved that (16) and (12) are equivalent. Therefore, since (10) and (7)–(9) are equivalent, Theorem $2'$ is a corollary to Theorem 2.

If the boundary conditions are not periodic but, say, the no-slip condition is satisfied, then transition to the abstract form is complicated. The difficulties result from the fact that the resolvent of the Laplace operator does not commute with the derivatives with respect to the spatial variables. Nevertheless, the transition is possible if we use, for instance, the results of Solonnikov in [7, 8].

It is not difficult to show that a classical (strong) solution to (16) must be trivial, but Theorem $2'$ deals with a generalized solution. Therefore, Theorem $2'$ does not solve the well-known problem of strong solvability, but reduces it to another problem.

REMARK 3. For the system of Navier–Stokes equations we can impose for $t = 0$ some nonzero Cauchy initial conditions. Then the problem of global strong solvability becomes more general than that under consideration. However, by the local existence theorem it reduces easily to the case of the zero Cauchy initial conditions. Let us verify that for the abstract problem (1). Take some infinitely smooth function $\varphi(t)$ such that $\varphi(t) = 0$ for $0 \le t < \varepsilon$, and $\varphi(t) = 1$ for $t \ge 2\varepsilon$, where ε is some small number. Suppose that (1) in [1] has a solution $u(t)$ on $(0, 4\varepsilon)$ such that $|B(u,u)|_H, |u' + Au|_H \in L_2(0, 4\varepsilon)$. Put $v = \varphi(t)u(t)$. For $v(t)$ we obtain $v(0) = 0$ and

$$v' + Av + B(v,v) = \begin{cases} (\varphi u)' + A\varphi u + \varphi^2 B(u,u) & \text{for } 0 < t \le 2\varepsilon, \\ f(t) & \text{for } t > 2\varepsilon. \end{cases}$$

Therefore, the general case reduces to the particular case.

REMARK 4. If it is known that a problem of type $(*)$ from Remark 3.1 in [1] is not globally strongly solvable then the type A theorems yield the existence of a solution to the system of the type (4) or $(4')$ in [1]. This sort of application of the type A theorems is useful even in the finite-dimensional case. Applying Theorem 1 to the scalar equation $y' + y^2 = f(t)$, $y(0) = 0$, we deduce the existence of a generalized solution to the boundary value problem $(4'')$ in [1].

A brief summary of this article is published in [9]; also see [10].

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