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PARTIAL DIFFERENTIAL EQUATIONS

Continuous Dependence of the Solution of a Parabolic Equation in a Hilbert Space on the Parameters and Initial Data

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*Dedicated to the seventieth birthday
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Abstract—We present classes of equations for which weak a priori estimates hold and prove the global strong solvability of one system of equations.

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INTRODUCTION

Let H be a separable real Hilbert space, and let A be a self-adjoint operator in H such that $A \geq E$. We assume that H is the space of functions defined on some locally compact set and the operator A takes real elements to real ones. By $C_A^\infty[0,a]$ $(a > 0)$ we denote the space of functions ranging in H, which, together with all of their derivatives, belong to $D(A^n)$ for any $n = 1, 2, \ldots$ [Here and throughout the following, $D(\cdot)$ stands for the domain of an operator.]

The completion of $C_A^\infty[0,a]$ in the norm

$$\|u\|_{H_\beta[0,a]} = \left(\int_0^a |A^\beta u(t)|^2 \, dt \right)^{1/2} \qquad (\beta \in (-\infty, \infty)) \tag{B.1}$$

is denoted by $H_\beta[0,a]$. The inner products in H and $H_\beta[0,a]$ are denoted by $\langle \cdot\,, \cdot \rangle$ and $\langle \cdot\,, \cdot \rangle_{\beta,a}$, respectively. The symbol a in the notation of the inner product $\langle \cdot\,, \cdot \rangle_{\beta,a}$ is sometimes omitted if this does not lead to a misunderstanding. The norm and the absolute value are denoted by $|\cdot|$; sometimes, to avoid misunderstanding, we denote the norm by $\|\cdot\|$.

In the present paper, we consider the Cauchy problem

$$u_t + Au + \theta(t)B(\lambda, u) = f(\lambda, t), \qquad u(0) = 0, \qquad 0 \leq t \leq a. \tag{B.2}$$

In the following, without loss of generality, we assume that $a = 1$. Here $\theta(t)$ is a continuous function on $[0,a]$, and $B(\lambda, \cdot)$ is a nonlinear operator analytically depending on a parameter $\lambda \in G = \{\lambda : \, |\lambda| < 1\}$. Note that, in problem (B.2), one can consider the general case in which the nonlinear term has the form $B(t, \lambda, u)$ rather than $\theta(t)B(\lambda, u)$. One can also consider the case in which A is the generator of a strongly continuous analytic semigroup; in this case, fractional powers of A are understood in accordance with [1, p. 337 of the Russian translation]. To simplify the exposition, we consider the case in which $B(t, \lambda, u)$ has the form $\theta(t)B(\lambda, u)$, $A = A^*$, and $A \geq E$.

Definition. Let β be a real number, and let $f(\cdot) \in H_\beta[0,1]$. A solution $u(\cdot)$ of problem (B.2) is said to be β-*strong* if $u' + Au \in H_\beta[0,1]$. If the solution of problem (B.2) is β-strong for any $f(\cdot) \in H_\beta[0,1]$, then problem (B.2) is said to be β-strongly solvable in the whole.

We are interested in problems of strong solvability of problem (B.2), since, as a rule, the presence of the strong solvability permits one to prove the uniqueness theorem and use perturbation theory, which is very important in applications.

We are also interested in problems in which the presence of a weak a priori estimate permits one to obtain the strong solvability, because one often deals with equations for whose solutions a weak a priori estimate can readily be obtained. Examples of equations for which a weak estimate is easy to obtain include the system of Navier–Stokes equations and some equations of magnetic gas and fluid dynamics.

In Sections 4 and 5, we present classes of equations for which weak a priori estimates hold and prove the global strong solvability of one system of equations.

We subject $B(\lambda, u)$ to a condition that expresses the fact that $B(\lambda, \cdot)$ is subordinate to the operator A and means the continuous dependence on λ and u whenever u belongs to $D(A)$.

Assumption B.1. Suppose that there exist constants $\beta \in (-\infty, \infty)$, $\gamma \in (1/2, 1)$, and $\delta \in (0, 1]$ such that the following conditions are satisfied.

(a) If $\lambda, \mu \in \bar{G} = \{\lambda : \ |\lambda| \le 1\}$ and $u, w \in D(A^{\beta+\gamma})$, then

$$\|A^\beta(B(\lambda, u) - B(\mu, u))\|_H \le |\lambda - \mu| \, \|A^{\beta+\gamma} u\|_H \varphi_1(\|A^{\beta+\gamma-1/2} u\|_H),$$

where $\varphi_1(\cdot)$ is a continuous nondecreasing function on $[0, +\infty)$.

(b) If $\lambda \in \bar{G}$ and $u, w \in D(A^{\beta+\gamma})$, then

$$B(\lambda, u + w) - B(\lambda, u) = B_0(\lambda, u)w + B_1(\lambda, u, w),$$

where the operator $B_0(\lambda, u)w$ acts as a linear operator on w for each u, and $B_1(\lambda, u, w)$ is a nonlinear operator. These transformations satisfy the conditions

$$\begin{aligned} \|A^\beta B_0(\lambda, u)w\|_H &\le [\|A^{\beta+\gamma} w\|_H + \|A^{\beta+\gamma-1/2} w\|_H \|A^{\beta+\gamma} u\|_H] \varphi_2(\|A^{\beta+\gamma-1/2} u\|_H), \\ \|A^\beta B_1(\lambda, u, w)\|_H &\le [\|A^{\beta+\gamma} w\|_H + \|A^{\beta+\gamma-1/2} w\|_H \|A^{\beta+\gamma} u\|_H] \\ &\quad \times \|A^{\beta+\gamma-1/2} w\|_H^\delta \varphi_3(\|A^{\beta+\gamma-1/2} u\|_H, \|A^{\beta+\gamma-1/2} w\|_H), \end{aligned}$$

where $\varphi_2(\cdot)$ is a continuous nondecreasing function on $[0, +\infty)$, $\varphi_3(\cdot\,, \cdot)$ is a continuous function on $[0, +\infty) \times [0, +\infty)$ nondecreasing with respect to each of the variables, and both of these functions are independent of $\lambda, \mu \in \bar{G}$.

(c) $B(\lambda, 0) = 0$.

(d) The function $f(\lambda, t)$ is analytic with respect to $\lambda \in G$, and

$$\int_0^1 |A^\beta(f(\lambda, t) - f(\mu, t))|^2 \, dt \le T|\lambda - \mu|^2,$$

where T is a constant number.

Note that condition (c) in Assumption B.1 is not restrictive, but we write it out for convenience.

It readily follows from conditions (b) and (c) in Assumption B.1 that if $w \in D(A^{\beta+\gamma})$, then

$$\begin{aligned} \|A^\beta B(\lambda, w)\|_H &\le \varphi_2(0)\|A^{\beta+\gamma} w\|_H + \|A^{\beta+\gamma-1/2} w\|_H^\delta \|A^{\beta+\gamma} w\|_H \varphi_3(0, \|A^{\beta+\gamma-1/2} w\|_H) \\ &\equiv \|A^{\beta+\gamma} w\|_H \varphi_4(\|A^{\beta+\gamma-1/2} w\|_H), \end{aligned} \tag{B.3}$$

where β, γ, δ, $\varphi_2(\cdot)$, and $\varphi_3(\cdot\,, \cdot)$ are the same as in Assumption B.1.

Theorem B.1. *Let the conditions of Assumption* B.1 *be valid. In addition, suppose that* $f(t) \in H_\beta[0, 1]$, *and for some* $\lambda = \lambda_0 \in \bar{G} = \{\lambda : \ |\lambda| \le 1\}$, *the Cauchy problem* (B.2) *has a solution* $u(t)$ *such that*

$$\int_0^1 |A^{\beta+\gamma-1/2}(u' + Au)|^2 \, dt \equiv \|u' + Au\|^2_{H_{\beta+\gamma-1/2}} \le C < \infty. \tag{B.4}$$

Suppose that the estimate

$$\left[\int_0^1 |A^\beta \tilde f|^2\,dt + (T+1)|\lambda_0-\mu|^2\right] \le \frac{\varepsilon_0^2}{2} \tag{B.5}$$

holds for $\mu \in \bar G$, *where* $\varepsilon_0^2 \equiv 16^{-1}\varepsilon_1^2\varphi_1^{-2}(C+1)^{-1}$, *and* ε_1 *satisfies the inequality*

$$0 \le \varepsilon_1 \le [40(C+1)(1+C_1+C_2)C_3e^\xi\varphi_3(2C+2,1)]^{-1/\delta}.$$

Here

$$\xi \ge [40(1+C_1+C_2)C_3\varphi_2(2C+2)]^{2/(1-\gamma)},$$

where C *is the number occurring in* (B.4), $\varphi_1(\cdot)$, $\varphi_2(\cdot)$, $\varphi_3(\cdot\,,\cdot)$, δ, *and* T *are the functions and the number occurring in Assumption* B.1, $C_3 \equiv \sup_{0\le t\le 1}|\theta(t)|$, C_1, *and* C_2 *depend only on* γ *and are different from* $+\infty$ *for* $\gamma \in (1/2,1)$. *(Here* C_1 *and* C_2 *are the numbers occurring in Lemma* 2.2 *for* $\theta=\gamma$.*)*

Then the Cauchy problem

$$v_t + Av + \theta(t)B(\mu,v) = f(\mu,t) - \tilde f(t), \qquad v(0)=0, \qquad 0\le t\le 1, \tag{B.2$'$}$$

has a solution $v(t)$, *and this solution satisfies the estimates*

$$\|v'+Av\|^2_{H_\beta[0,1]} \le 8\varphi_4^2(C+1)(C+1)^2, \tag{B.6}$$

$$\|v'+Av\|^2_{H_{\beta+\gamma-1}[0,1]} \le 2C+1, \tag{B.6$'$}$$

$$\|v'-u'+Av-Au\|^2_{H_\beta[0,1]} \le 8\varepsilon_1^2. \tag{B.6$''$}$$

This theorem actually states that the solution of the Cauchy problem (B.2) continuously depends on the parameter λ and on $f \in H_\beta[0,1]$ in the solvability domain.

In the proof of the theorem, we have to consider a family of problems depending on a discrete parameter N. The constants in Theorem B.1 arise when estimating the solutions of systems depending on the parameter N. To emphasize the fact that the constants are independent of the parameter N, we preserve them in the form in which they appear.

By P_N we denote the orthogonal projection defined by the formula

$$P_N u = \int_0^N (dE_\lambda)u, \qquad N>1, \tag{B.7}$$

where E_λ is the spectral resolution of identity corresponding to the self-adjoint positive operator A.

We introduce the nonlinear operators

$$B_N(\lambda,u) = P_N B(\lambda, P_N u). \tag{B.7$'$}$$

In the forthcoming considerations, we use the following assumptions.

Assumption B.2. (a) If

$$\|f\|^2_{H_\beta[0,1]} \equiv \int_0^1 \|A^\beta f\|^2_H dt \le C_0 < \infty, \tag{B.8}$$

then the solution $u_N(\lambda,t)$ of the Cauchy problem

$$\frac{d}{dt}u + Au + \theta(t)B_N(\lambda,u) = P_N f(t), \qquad u(0)=0, \qquad 0\le t\le 1, \tag{B.9}$$

exists for any N and analytically depends on the parameter $\lambda \in G = \{\lambda : |\lambda| < 1\}$; moreover, the estimate

$$\sup_{N\geq 1} \sup_{\lambda\in\partial G} \int_0^1 \|A_N^\alpha u_N(\lambda, t)\|_H^2 \, dt \leq C_1 < \infty$$

holds for some $\alpha \in (-\infty, +\infty)$, $\alpha < \beta + 1/2$.

(b) For any $N \geq 1$, there exists a $\lambda_N \in \bar{G} = \{\lambda : |\lambda| \leq 1\}$ such that if condition (B.8) is satisfied, then the solution $u_N(\lambda_N, t)$ of problem (B.9) (with $\lambda = \lambda_N$) satisfies the estimate

$$\sup_{N\geq 1} \left[\int_0^1 \left\| A^\beta \frac{d}{dt} u_N(\lambda_N, t) \right\|_H^2 dt + \int_0^1 \|A^{\beta+1} u_N(\lambda_N, t)\|_H^2 \, dt \right] \leq C_1 < \infty.$$

Theorem B.2. *Let Assumptions* B.1 *and* B.2 *be satisfied. Then the following assertions hold.*

(a) *If* $f(\cdot) \in H_\beta[0,1]$, *then, for* $\varepsilon \in (0,1)$, *the solution* $u_N(\lambda, t)$ *of problem* (B.9) *satisfies the estimate*

$$\sup_{|\lambda|<1-\varepsilon} \sup_{N\geq 1} \int_0^1 \left(\left\| \frac{d}{dt} A^\beta u_N(\lambda_N, t) \right\|_H^2 + \|A^{\beta+1} u_N(\lambda_N, t)\|_H^2 \right) dt = C_\varepsilon < \infty.$$

(b) *If* $|\lambda| < 1$, *then all weak solutions of problem* (B.8) *weakly converge to a unique limit* $u(\lambda, t)$ *as* $N \to \infty$, *and this limit is a solution of problem* (B.2) *satisfying the condition*

$$\int_0^1 \left(\left\| \frac{d}{dt} A^\beta u(\lambda, t) \right\|_H^2 + \|A^{\beta+1} u(\lambda, t)\|_H^2 \right) dt < \infty.$$

(c) *If* $|\lambda| < 1$, *then the solution* $u(\lambda, t)$ *of problem* (B.2), *as well as* $\frac{d}{dt} u(\lambda, t)$ *and* $Au(\lambda, t)$, *continuously depends in the metric of* $H_\beta[0,1]$ *on the parameter* $\lambda \in G = \{\lambda : |\lambda| < 1\}$ *and on the right-hand side of problem* (B.2) *continuously changing in the metric of* $H_\beta[0,1]$.

This theorem claims that the presence of a weak estimate for the solution of problem (B.2) for all $\lambda \in \partial G$ and a strong estimate for at least one $\lambda \in \bar{G}$ results in the strong estimate for the solution of problem (B.2) for all λ lying inside the unit disk.

Remark B.1. Note that if we take $\widetilde{P_N} = e^{-A/N}$ instead of P_N in (B.7), then Theorems B.1 and B.2 remain valid provided that A is the infinitesimal operator of an analytic semigroup in a Hilbert space. In this case, the fractional powers of the operator A can be defined just as in [1, p. 357 of the Russian translation].

Instead of the Cauchy problem (B.2), one can consider the problem

$$u_t + Au + \theta(t) B(\lambda, u) = f(t), \qquad u(0) = \varphi, \qquad 0 \leq t \leq 1, \tag{B.2$''$}$$

where $\varphi \in D(A^{\beta+1/2})$ and $f(t) \in H_\beta[0,1]$, β being taken from Assumption B.1. If the local existence theorem holds, then such a problem can readily be reduced to problem (B.2).

The requirements of Assumption B.1 (i.e., the assumptions of Theorem B.1) are natural.

To justify Assumption B.2, below we consider an example of equations for which condition (a) in Assumption B.2 holds.

Let $B(u, v)$ be a bilinear operator (i.e., linear with respect to one of the arguments with the other argument being fixed) taking real elements u and v in H to a real element such that

$$\langle B(u, g), v \rangle_H = -\langle B(u, v), g \rangle_H. \tag{B.10}$$

This relation holds for the nonlinear part of the system of Navier–Stokes equations.

In Section 4, we consider the system

$$\begin{gathered} u_k' + Au_k(t) + \alpha_{sj}^{(k)}\theta(t)B(u_s, u_j) = f_k(t), \qquad k = 1, 2, \\ u_1(t)|_{t=0} = u_2(t)|_{t=0} = 0, \qquad 0 \le t \le 1, \end{gathered} \tag{B.11}$$

where the $\alpha_{sj}^{(l)}$ $(l, s, j = 1, 2)$ are real numbers and $f_1(t)$ and $f_2(t)$ are real functions ranging in H. In (B.11), repeated indices imply summation.

Set

$$\begin{gathered} d = \alpha_{21}^{(2)}\alpha_{12}^{(1)} - \alpha_{11}^{(2)}\alpha_{22}^{(1)} \ne 0, \\ a_{11} = d^{-1}[-\alpha_{21}^{(2)}(\alpha_{12}^{(2)} - \alpha_{11}^{(1)}) + \alpha_{11}^{(2)}(\alpha_{22}^{(2)} - \alpha_{21}^{(1)})], \\ a_{22} = d^{-1}[-\alpha_{22}^{(1)}(\alpha_{12}^{(2)} - \alpha_{11}^{(1)}) + \alpha_{12}^{(1)}(\alpha_{22}^{(2)} - \alpha_{21}^{(1)})]. \end{gathered} \tag{B.12}$$

Theorem B.3. *If* $a_{11}a_{22} > 1$ *and* $a_{11} > 0$, *then the solution of the Cauchy problem* (B.11) *satisfies the a priori estimate*

$$\sup_{0\le t\le 1}(\|u_1(t)\|_H^2 + \|u_2(t)\|_H^2) + \int_0^1 (|\sqrt{A}\,u_1(\eta)|^2 + |\sqrt{A}\,u_2(\eta)|^2)\,d\eta$$
$$\le 64\frac{(a_{11} + a_{22})^2 \int_0^1 (|f_1(\eta)|^2 + |f_2(\eta)|^2)\,d\eta}{[\min(a_{11} - a_{22}^{-1}, a_{22} - a_{11}^{-1})]^2}. \tag{B.13}$$

Following [2, p. 88 of the Russian translation] (for the case in which A^{-1} is a compact operator), from this theorem, one can find that problem (B.11) has a weak solution satisfying the estimate (B.13).

In system (B.11), we assume that the $\alpha_{sj}^{(l)}$ $(l, s, j = 1, 2)$ are functions of $\varphi \in [0, 2\pi[$ on $\partial G = \{z : \ |z| = 1\}$ (φ is the same as in the representation $z = e^{i\varphi}$) and a_{11} and a_{22} satisfy the condition $a_{11}a_{22} > 1 + \delta$, $\delta > 0$, for each $\varphi \in [0, 2\pi[$, where a_{11} and a_{22} are taken from (B.12),

$$f_1(t) = g_1(\varphi)f(t), \qquad f_2(t) = g_2(\varphi)f(t), \tag{B.14}$$

and $g_1(\cdot)$ and $g_2(\cdot)$ are real functions on ∂G.

Let

$$b \equiv (b_{sj})_{s,j=1}^2, \qquad \gamma = (\gamma_{sj})_{s,j=1}^2 = \frac{1}{\Delta}\begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}, \tag{B.15}$$

where $\Delta \equiv b_{11}b_{22} - b_{12}b_{21} \ne 0$ and the b_{sj} $(s, j = 1, 2)$ are complex-valued functions on ∂G.

For $z \in \partial G$ in system (B.11), we set

$$u_k = \gamma_{k1}v_1 + \gamma_{k2}v_2, \qquad k = 1, 2.$$

Then from (B.11), (B.14), and (B.15), we obtain the system

$$\begin{gathered} \gamma_{kl}\frac{d}{dt}v_l + \gamma_{kl}Av_l(t) + \alpha_{sj}^{(k)}\gamma_{sm}\gamma_{jn}B(v_m, v_n) = g_k(\varphi)f(t), \qquad k = 1, 2, \\ v_l(t)|_{t=0} = 0, \qquad 0 \le t \le 1, \end{gathered}$$

which can be reduced to the system of equations

$$\frac{d}{dt}v_l + Av_l + b_{lk}\alpha_{sj}^{(k)}\gamma_{sm}\gamma_{jn}B(v_m, v_n) = b_{lk}g_k(\varphi)f(t) \qquad (l = 1, 2). \tag{B.11$'$}$$

We choose $\alpha_{sj}^{(l)}$, b_{sj}, $g_1(\cdot)$, and $g_2(\cdot)$ $(l, s, j = 1, 2)$ so as to ensure that the functions $b_{lk}\alpha_{sj}^{(k)}\gamma_{sm}\gamma_{jn}$ and $b_{lk}g_k(\varphi)$ $(l, m, n = 1, 2)$ are the values on ∂G of analytic functions in G. In Section 4, we show that such a choice is possible. Therefore, for the truncated systems corresponding to system (B.11)$'$, from Theorem B.3, we obtain a priori estimates independent of N (see Theorem B.3, Section 4). Consequently, system (B.11)$'$ is an example for which condition (a) in Assumption B.2 holds (see Theorem 4.1 and Theorem 5.1 in Section 5).

In Section 6, we reduce the Cauchy problem for the well-known Navier–Stokes equations (see [3, p. 73]) to the form (B.2). Therefore, it becomes possible to use abstract results for the system of the Navier–Stokes equations. Although system (5.8) is "similar" to an abstract form of the Navier–Stokes equations, Theorem 5.1 does not solve the problem of the existence of a global strong solution of the Navier–Stokes equations.

1. SOME THEOREMS ON OPERATOR-VALUED ANALYTIC FUNCTIONS

Here we present some estimates for operators, which are proved on the basis of interpolation ideas going back to Riesz.

Let H be a separable Hilbert space, let $A \geq 0$ be a self-adjoint operator with compact resolvent, let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal system of eigenvectors, and let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of the corresponding eigenvalues. We have $\lambda_n \geq 0$ for any $n = 1, 2, \ldots$ We number λ_n in nondescending order with regard of multiplicities. The expansion

$$c = \sum_{n=1}^{\infty} c_n e_n$$

holds for arbitrary $c \in H$.

By definition,

$$A^{\alpha}c = \sum_{n=1}^{\infty} c_n \lambda_n^{\alpha} e_n, \qquad |A^{\alpha}c|^2 = \sum_{n=1}^{\infty} |c_n|^2 \lambda_n^{2\alpha}$$

for $\alpha \geq 0$.

Lemma 1.1. *Let $0 \leq \beta < \gamma$ and $0 \leq \beta \leq \alpha \leq \gamma$. Then*

$$|A^{\alpha}c|^2 \leq 2|A^{\gamma}c|^{2(\alpha-\beta)/(\gamma-\beta)}|A^{\beta}c|^{2(\gamma-\alpha)/(\gamma-\beta)}. \tag{1.1}$$

Proof. For $\mu > 0$, we have

$$\begin{aligned}
|A^{\alpha}c|^2 &= \sum_{n=1}^{\infty} |c_n|^2 \lambda_n^{2\alpha} = \sum_{\lambda_n \leq \mu} |c_n|^2 \lambda_n^{2\alpha} + \sum_{\lambda_n > \mu} |c_n|^2 \lambda_n^{2\alpha} \\
&\leq \left(\sum_{n=1}^{\infty} |c_n|^2 \lambda_n^{2\beta}\right) \mu^{2(\alpha-\beta)} + \left(\sum_{n=1}^{\infty} |c_n|^2 \lambda_n^{2\gamma}\right) \mu^{2(\alpha-\gamma)} \\
&= \mu^{2(\alpha-\beta)}|A^{\beta}c|^2 + \mu^{2(\alpha-\gamma)}|A^{\gamma}c|^2.
\end{aligned}$$

Take

$$\mu^{2(\gamma-\beta)} = \mu^{2(\alpha-\beta)-2(\alpha-\gamma)} = |A^{\beta}c|^{-2}|A^{\gamma}c|^2.$$

Then we obtain the inequality

$$|A^{\alpha}c|^2 \leq 2|A^{\beta}c|^2 \left(\frac{|A^{\gamma}c|^2}{|A^{\beta}c|^2}\right)^{2(\alpha-\gamma)/(2(\gamma-\beta))} = 2|A^{\gamma}c|^{2(\alpha-\beta)/(\gamma-\beta)}|A^{\beta}c|^{2(1-(\alpha-\beta)/(\gamma-\beta))},$$

which implies the estimate (1.1). The proof of the lemma is complete.

This lemma remains valid without the assumption that the resolvent is compact, i.e., when the spectrum of the self-adjoint nonnegative operator A is not discrete; namely, the following assertion holds.

Lemma 1.1′. *Let A be a self-adjoint operator in H. If $0 \le \beta < \gamma$ and $0 \le \beta \le \alpha \le \gamma$, then the estimate* (1.1) *holds.*

To prove this lemma, instead of the Fourier expansion in eigenvectors, one should use the Stieltjes integral associated with the spectral resolution of identity corresponding to the self-adjoint operator A.

Now let $\psi(z)$ be an analytic function with values in H. For $\psi(z)$, we have the representation

$$\psi(z) = \sum_{n=1}^{\infty} \psi_n z^n, \tag{1.2}$$

where the ψ_n are elements of H. Let us compute $|A^\alpha \psi_n|$.

If we take the inner product of relation (1.2) by $(A^{2\alpha}\psi_n)z^n$ (in the sense of the complex Hilbert space), integrate the resulting relation over a circle of radius $r \in (0,1)$, and take φ as the integration variable from the representation $z = re^{i\varphi}$, then we obtain

$$\int_0^{2\pi} \langle A^\alpha \psi(z), A^\alpha \psi_n\rangle r^n e^{-i\varphi}\, d\varphi = |A^\alpha \psi_n|^2 r^{2n}.$$

Hence we obtain the inequality

$$r^n |A^\alpha \psi_n|^2 \le \int_0^{2\pi} |A^\alpha \psi(re^{i\varphi})|\, |A^\alpha \psi_n|\, d\varphi,$$

or

$$|A^\alpha \psi_n| \le r^{-n} \int_0^{2\pi} |A^\alpha \psi(re^{i\varphi})|\, d\varphi. \tag{1.3}$$

Lemma 1.2. *Let $0 < \beta < \gamma$, let $0 < \beta \le \alpha \le \gamma$, and let $\psi(z)$ be the function that takes values in H, is analytic in the disk $G_{r_i} = \{z : \ |z| < r_i\}$ $(i = 1, 2)$, and satisfies the conditions*

$$\int_0^{2\pi} |A^\beta \psi(r_1 e^{i\varphi})|\, d\varphi = c_1 < \infty, \qquad \int_0^{2\pi} |A^\gamma \psi(r_2 e^{i\varphi})|\, d\varphi = c_2 < \infty.$$

Then the inequality

$$\sup_{|z|\le r} |A^\alpha \psi(z)| \le \frac{c_0}{1 - r r_1^{-(\gamma-\alpha)/(\gamma-\beta)} r_2^{-(\alpha-\beta)/(\gamma-\beta)}}$$

holds for $r_2 < r < r_1$, where $c_0 = 2c_1^{2(\gamma-\alpha)/(\gamma-\beta)} c_2^{2(\alpha-\beta)/(\gamma-\beta)}$.

Proof. The representation

$$\psi(z) = \sum_{n=0}^{\infty} \psi_n z^n$$

holds in the disk G_{r_i}. By using (1.3), hence we obtain the inequalities

$$|A^\beta \psi_n| \le c_1 r_1^{-n}, \qquad |A^\gamma \psi_n| \le c_2 r_2^{-n},$$

which, together with Lemma 1.1, imply that

$$\begin{aligned}|A^\alpha\psi_n|^2 &\le 2(c_1r_1^{-n})^{2(\gamma-\alpha)/(\gamma-\beta)}(c_2r_2^{-n})^{2(\alpha-\beta)/(\gamma-\beta)}\\ &\le 2c_1^{2(\gamma-\alpha)/(\gamma-\beta)}c_2^{2(\alpha-\beta)/(\gamma-\beta)}(r_1^{-2(\gamma-\alpha)/(\gamma-\beta)}r_2^{-2(\alpha-\beta)/(\gamma-\beta)})^n.\end{aligned}$$

By virtue of the last inequality, we have

$$r^n|A^\alpha\psi_n| \le c_0(rr_1^{-(\gamma-\alpha)/(\gamma-\beta)}r_2^{-(\alpha-\beta)/(\gamma-\beta)})^n;$$

therefore,

$$\begin{aligned}\sup_{|z|\le r}|A^\alpha\psi(z)| &\le \sum_{n=0}^{\infty} r^n|A^\alpha\psi_n| \le c_0\sum_{n=0}^{\infty}(rr_1^{-(\gamma-\alpha)/(\gamma-\beta)}r_2^{-(\alpha-\beta)/(\gamma-\beta)})^n\\ &\le \frac{c_0}{1-rr_1^{-(\gamma-\alpha)/(\gamma-\beta)}r_2^{-(\alpha-\beta)/(\gamma-\beta)}}.\end{aligned}$$

The proof of the lemma is complete.

2. SOME ESTIMATES FOR THE SOLUTION OF A NONLINEAR PARABOLIC EQUATION

Here we derive some estimates for the solution of the parabolic equation (B.11) for the case in which A and $B(\lambda,\cdot)$ are continuous operators. The assumption that A and $B(\lambda,\cdot)$ are continuous makes it unnecessary to satisfy the condition that the elements belong to the domain in the consideration of various expressions related with A and $B(\lambda,\cdot)$. But since it is necessary to pass to the case in which these transformations are unbounded, we should point out estimates (with exact orders) for the constants that appear in the course of considerations and are to be used below for the passage to the case of unbounded transformations A and $B(\lambda,\cdot)$.

Thus, let $A \ge E$ be a bounded self-adjoint operator in a separable Hilbert space H, and let $B(\lambda,\cdot)$ be a continuous nonlinear operator analytically depending on the parameter $\lambda \in G$ and continuously depending on $\lambda \in \bar{G} = \{\lambda : \ |\lambda| \le 1\}$. The time interval on which the problem is considered can be equal to $(0,1]$; i.e., $a = 1$.

In this section, we use Assumption B.1 with the same constraints for the functions $\varphi_1(\cdot)$, $\varphi_2(\cdot)$, $\varphi_3(\cdot,\cdot)$, and $\varphi_4(\cdot)$ and the following assumption.

Assumption B.2′. (a) If $f(\cdot) \in H_\beta[0,1]$, where β is the same as in Assumption B.1, then the solution of the Cauchy problem

$$u_t + Au + B(\lambda,u) = f, \qquad u(0) = 0, \qquad 0 \le t \le 1, \tag{2.1}$$

exists for some $\alpha \in (-\infty,+\infty)$, depends analytically on the parameter $\lambda \in G$, and satisfies the estimate

$$\sup_{\lambda\in\partial G}\sup_{0\le t\le 1}|A^{\alpha-1/2}u(t)|^2 + \sup_{\lambda\in\partial G}\int_0^1 |A^\alpha u(t)|^2\,dt \le C_0^2(\|A^\alpha f\|_{H[0,1]}), \tag{2.2}$$

where $C_0(\cdot)$ is a continuous function on $[0,+\infty)$.

(b) If $f(\cdot) \in H_\beta[0,1]$, then, for some $\lambda_0 \in \bar{G}$, the solution of problem (2.1) exists and satisfies the estimate

$$\int_0^1 \left(\left\|\frac{d}{dt}A^\beta u(\lambda_0,t)\right\|_H^2 + \|A^{\beta+1}u(\lambda_0,t)\|_H^2\right)dt \le C_1(\|A^\alpha f\|_{H[0,1]}), \tag{2.2$'$}$$

where $C_1(\cdot)$ is a continuous nondecreasing function on $[0,+\infty)$.

Assumptions B.1 and B.2′ are not used in Lemmas 2.1 and 2.2.

Lemma 2.1. *Suppose that* $A \geq E$, $\theta \in (-\infty, +\infty)$, $\xi \geq 0$, $u(t)$ *is a continuous function with values in* H, *and* $\frac{d}{dt}u(t) \in H[0,1]$. *Then one has*

$$\|(A+\xi E)^{\theta}u\|^2(\eta)\Big|_0^t + \int_0^t \|(A+\xi E)^{\theta+1/2}u(\eta)\|^2\,d\eta \leq \int_0^t \left\|(A+\xi E)^{\theta-1/2}\left(\frac{d}{d\eta}+A+\xi E\right)u(\eta)\right\|^2 d\eta,$$

$$\int_0^t \left\|(A+\xi E)^{\theta-1/2}\frac{d}{d\eta}u(\eta)\right\|^2 d\eta \leq 4\int_0^t \left\|(A+\xi E)^{\theta-1/2}\left(\frac{d}{d\eta}+A+\xi E\right)u(\eta)\right\|^2 d\eta - 2\|(A+\xi E)^{\theta}u\|^2(\eta)\Big|_0^t.$$

Proof. The first inequality in the lemma follows from the relations

$$\begin{aligned}
&\|(A+\xi E)^{\theta}u\|^2(\eta)\Big|_0^t + 2\int_0^t \|(A+\xi E)^{\theta+1/2}u(\eta)\|^2\,d\eta \\
&= 2\int_0^t \langle u' + (A+\xi E)u, (A+\xi E)^{2\theta}u\rangle\,d\eta \\
&= 2\int_0^t \langle (A+\xi E)^{\theta-1/2}(u' + (A+\xi E)u), (A+\xi E)^{\theta+1/2}u\rangle\,d\eta \\
&\leq \int_0^t |(A+\xi E)^{\theta-1/2}(u'+(A+\xi E)u)|^2\,d\eta + \int_0^t |(A+\xi E)^{\theta+1/2}u(\eta)|^2\,d\eta.
\end{aligned}$$

The second inequality can readily be derived from the first one and from the triangle inequality. The proof of the lemma is complete.

Lemma 2.2. *Let* $1/2 < \theta < 1$. *Then*

$$\int_0^1 |(A+\xi E)^{\theta}T_\xi g|^2(\eta)\,d\eta \leq \frac{C_1^2}{(\xi+1)^{1-\theta}}\int_0^1 |g(\eta)|^2\,d\eta \qquad \forall g \in DT_\xi,$$

$$\sup_{0\leq t\leq 1} |(A+\xi E)^{\theta-1/2}T_\xi g|^2(\eta)\,d\eta \leq \frac{C_2^2}{(\xi+1)^{1-\theta}}\int_0^1 |g(\eta)|^2\,d\eta \qquad \forall g \in DT_\xi,$$

where $C_1 \equiv C_1(\theta)$ *and* $C_2 \equiv C_2(\theta)$ *are positive numbers that depend only on* θ *and*

$$T_\xi g = \int_0^t e^{-(A+\xi)(t-\eta)}g(\eta)\,d\eta, \qquad g \in DT_\xi.$$

Proof. We have

$$\begin{aligned}|(A+\xi E)^{\theta}(T_{\xi}g)(t)| &\le (\xi+1)^{-(1-\theta)/2}|(A+\xi E)^{\theta+(1-\theta)/2}(T_{\xi}g)(t)| \\ &\le (\xi+1)^{-(1-\theta)/2}\left|\int\limits_0^t [(A+\xi E)(t-\eta)]^{(1+\theta)/2}\frac{e^{-(A+\xi)(t-\eta)}g(\eta)}{(t-\eta)^{(1+\theta)/2}}\,d\eta\right| \\ &\le C\left(\frac{\theta}{2}\right)(\xi+1)^{-(1-\theta)/2}\left|\int\limits_0^t \frac{|g(\eta)|}{|t-\eta|^{(1+\theta)/2}}\,d\eta\right|.\end{aligned}$$

Here we have used the inequality

$$|((A+\xi E)\tau)^{(1+\theta)/2}e^{-(A+\xi)\tau}| \le \sup_{\chi>0}\chi^{(1+\theta)/2}e^{-\chi} \le C\left(\frac{1+\theta}{2}\right) < \infty. \tag{2.3}$$

We continue the function $|g(\eta)|$ outside the interval $[0,1]$ by zero. Then from (2.3), we have

$$|(A+\xi E)^{\theta}(T_{\xi}g)(t)| \le C\left(\frac{\theta}{2}\right)(\xi+1)^{-(1-\theta)/2}\int\limits_{-\infty}^{+\infty}\frac{e^{-(1/2)|t-\eta|}|g(\eta)|}{|t-\eta|^{(1+\theta)/2}}\,d\eta. \tag{2.3$'$}$$

Now, by using estimates for operators invariant under shifts [4, p. 52 of the Russian translation], we obtain the first inequality of the lemma. Note that if $0<\theta<1$, then the kernel $|t-\eta|^{-(1+\theta)/2}$ is a kernel with a weak singularity; therefore, the derivation of the first inequality in the lemma from (2.3)$'$ is well known in the theory of integral operators.

Further,

$$\begin{aligned}|(A+\xi E)^{\theta-1/2}T_{\xi}g|(t) &= \left|\int\limits_0^t e^{-(A+\xi)(t-\eta)}(A+\xi E)^{\theta-1/2}g(\eta)\,d\eta\right| \\ &\le (\xi+1)^{-(1-\theta)/2}\int\limits_0^t \left|e^{-(A+\xi)(t-\eta)}(A+\xi E)^{\theta-1/2-\theta/2+1/2}g(\eta)\right|\,d\eta \\ &\le (\xi+1)^{-(1-\theta)/2}\int\limits_0^t \left|e^{-(A+\xi)(t-\eta)}[(A+\xi E)(t-\eta)]^{\theta/2}\frac{g(\eta)}{(t-\eta)^{\theta/2}}\right|\,d\eta \\ &\le C\left(\frac{\theta}{2}\right)(\xi+1)^{-(1-\theta)/2}\int\limits_0^t \frac{|g(\eta)|}{|t-\eta|^{\theta/2}}\,d\eta \\ &\le C\left(\frac{\theta}{2}\right)\frac{1}{(\xi+1)^{(1-\theta)/2}}\left(\int\limits_0^t |g(\eta)|^2\,d\eta\right)^{1/2}\left(\int\limits_0^t \frac{1}{(t-\eta)^{\theta}}\,d\eta\right)^{1/2}.\end{aligned}$$

Here we have used inequality (2.3). This implies the second inequality in the lemma. The proof of the lemma is complete.

Lemma 2.3. *Let Assumption* B.1 *be satisfied, let* $f \in H_{\beta}[0,1]$ *(*β *is the same as in Assumption* B.1*), and, in addition, let the Cauchy problem* (B.2) *with some* $\lambda \in \bar{G} = \{\lambda : \ |\lambda| \le 1\}$ *and with* $a=1$ *has a solution* $u(t)$ *such that*

$$\int\limits_0^1 |A^{\beta+\gamma-1}(u'+Au)|^2\,dt \le C < \infty, \tag{2.4}$$

where β *and* γ *are the numbers from Assumption* B.1.

Let condition (B.5) *hold for* $\tilde{f}$ *and* $\mu \in \bar{G}$. *Then the solution* $v(t)$ *of the Cauchy problem* (B.2)$'$ *satisfies inequalities* (B.2) *and* (B.6)$'$.

Proof. The operators A and $B(\cdot\,,\cdot)$ are continuous in the space H, and Assumption B.1 holds. Therefore, the equations for $u(t)$ and $v(t)$ have solutions. From the equation for $u(t)$, we subtract the equations for $v(t)$:

$$w' + Aw + \theta(t)[B(\lambda, u) - B(\mu, v)] = \tilde{f}(t), \qquad w(0) = 0, \qquad 0 \le t \le 1, \tag{2.5}$$

where $w = u - v$. Set $s(t) = e^{-\xi t} w(t)$, where $\xi > 0$. For $s(t)$, from (2.5), we obtain the equation

$$\begin{aligned} &s' + (A + \xi E)s + e^{-\xi t}\theta(t)[B(\lambda, u) - B(\mu, u - e^{\xi t}s(t))] = e^{-\xi t}\tilde{f}(t), \\ &s(0) = 0, \qquad 0 \le t \le 1. \end{aligned} \tag{2.5$'$}$$

Set $s' + (A + \xi E)s = A^{-\beta} g$. Then from (2.5)$'$ for g, we obtain

$$g(t) = (Fg)(t) \equiv e^{-\xi t} A^{\beta}\{\theta(t)[-B(\lambda, u) + B(\mu, u - e^{\xi t} T_\xi A^{-\beta} g)] + \tilde{f}\}, \tag{2.6}$$

where

$$T_\xi A^{-\beta} g = \int_0^t e^{-(A+\xi)(t-\eta)} A^{-\beta} g(\eta)\, d\eta. \tag{2.6$'$}$$

For arbitrary $g_1, g_2 \in H[0, a]$, we have

$$\begin{aligned} \|Fg_1 - Fg_2\|_H &= \|e^{-\xi t}\theta(t)A^{\beta}[B(\mu, u - e^{\xi t}T_\xi A^{-\beta} g_1) - B(\mu, u - e^{\xi t}T_\xi A^{-\beta} g_2)]\|_H \\ &= \|e^{-\xi t}\theta(t)A^{\beta}[B(\mu, w_1) - B(\mu, w_1 + w_2)]\|_H; \end{aligned} \tag{2.7}$$

here $w_1 = u - e^{\xi t} T_\xi A^{-\beta} g_1$ and $w_2 = e^{\xi t} T_\xi A^{-\beta}(g_1 - g_2)$.

By condition (b) in Assumption B.1,

$$B(\mu, w_1 + w_2) - B(\mu, w_1) = B_0(\mu, w_1)w_2 + B_1(\mu, w_1, w_2).$$

By taking into account the last relation, from (2.7), we obtain the estimate

$$\|Fg_1 - Fg_2\|_H \le \|e^{-\xi t}\theta(t)A^{\beta}B_0(\mu, w_1)w_2\|_H + \|e^{-\xi t}\theta(t)A^{\beta}B_1(\mu, w_1, w_2)\|_H.$$

By estimating the right-hand side with regard of Assumption B.1, we obtain

$$\begin{aligned} \|Fg_1 - Fg_2\|_H \le{}& C_3\varphi_2(\|A^{\beta+\gamma-1/2}w_1\|_H)[\|A^{\beta+\gamma}T_\xi A^{-\beta}(g_1 - g_2)\|_H \\ &+ \|A^{\beta+\gamma-1/2}T_\xi A^{-\beta}(g_1 - g_2)\|_H \|A^{\beta+\gamma}w_1\|_H] \\ &+ C_3\|A^{\beta+\gamma-1/2}w_2\|_H^{\delta}[\|A^{\beta+\gamma}w_2\|_H + \|A^{\beta+\gamma-1/2}w_2\|_H\|A^{\beta+\gamma}w_1\|_H] \\ &\times \varphi_3(\|A^{\beta+\gamma-1/2}w_1\|_H, \|A^{\beta+\gamma-1/2}w_2\|_H); \end{aligned} \tag{2.7$'$}$$

here $C_3 \equiv \sup_{0\le t\le 1}|\theta(t)|$. From Lemma 2.1 and from the representation of w_1 via u and g_1 [see (2.7)], we have

$$\begin{aligned} &\|A^{\beta+\gamma-1/2}w_1\|_H^2 + \int_0^t |A^{\beta+\gamma}w_1(\eta)|^2\, d\eta \\ &\quad\le \int_0^t \left| A^{\beta+\gamma-1}\left(\frac{d}{d\eta} + A\right)\left[u - e^{\xi\eta}\int_0^\eta e^{-(A+\xi)(\eta-\tau)}A^{-\beta}g_1(\tau)\, d\tau\right]\right|^2 d\eta \\ &\quad= \int_0^t \left| A^{\beta+\gamma-1}\left(\frac{d}{d\eta} + A\right)\left[u - \int_0^\eta e^{-A(\eta-\tau)}A^{-\beta}e^{\xi\tau}g_1(\tau)\, d\tau\right]\right|^2 d\eta \\ &\quad\le \int_0^t \left| A^{\beta+\gamma-1}\left(\frac{d}{d\eta} + A\right)u\right|^2 d\eta + \int_0^t |A^{\gamma-1}e^{\xi\tau}g_1(\tau)|^2\, d\tau. \end{aligned}$$

The elements g_1 and g_2 are chosen from the ball

$$M = \left\{ g : \int_0^t |g(\tau)|^2 \, d\tau \le \varepsilon_1^2 e^{-2\xi} \right\}, \tag{2.8}$$

where ε_1 is a small number in the interval $(0, 1/2)$. Then the last inequality and the assumption of the lemma imply the estimate

$$\|A^{\beta+\gamma-1/2} w_1\|_H^2 + \int_0^t |A^{\beta+\gamma} w_1(\eta)|^2 \, d\eta \le 2(C+1), \tag{2.9}$$

where C is the same as in $(B.6)'$.

Let us estimate w_2 by using Lemma 2.1 and the expression $(2.6)'$:

$$\begin{aligned} &\|A^{\beta+\gamma-1/2} w_2\|_H^2 + \int_0^t |A^{\beta+\gamma} w_2(\eta)|^2 \, d\eta \\ &\qquad \le \int_0^t \left| A^{\beta+\gamma-1} \left(\frac{d}{d\eta} + A \right) e^{\xi\eta} T_\xi A^{-\beta} (g_1 - g_2) \right|^2 d\eta \\ &\qquad \le \int_0^t |A^{\gamma-1} e^{\xi\eta} (g_1 - g_2)|^2 \, d\eta \le e^{2a\xi} \int_0^a |(g_1 - g_2)|^2 \, d\eta. \end{aligned} \tag{2.9$'$}$$

By using Lemma 2.2 for $\theta = \gamma$, the estimates (2.9) and $(2.9)'$, and the monotonicity of $\varphi_2(\cdot)$, we obtain

$$\begin{aligned} S_1 &\equiv \int_0^1 \{\varphi_2(\|A^{\beta+\gamma-1/2} w_1\|_H)[|A^\gamma T_\xi (g_1 - g_2)| + |A^{\beta+\gamma} w_1| \, |A^{\gamma-1/2} T_\xi (g_1 - g_2)|]\}^2 \, d\eta \\ &\le (1 + C_1 + C_2)^2 \frac{2\varphi_2^2(2C+2)}{(1+\xi)^{1-\gamma}} \\ &\quad \times \left[\int_0^1 |g_1 - g_2|^2 \, d\eta + \left(\int_0^1 |g_1 - g_2|^2 \, d\eta \right) \int_0^1 \|A^{\beta+\gamma} w_1\|^2 \, d\eta \right], \end{aligned} \tag{2.10}$$

where C_1 and C_2 are the same as in Lemma 2.2 for $\theta = \gamma$.

To estimate $A^{\beta+\gamma} w_1$, we use inequality (2.9) again; then we obtain

$$S_1 \le 8(1+C)(1 + C_1 + C_2)^2 \frac{4\varphi_2^2(2C+2)}{(1+\xi)^{1-\gamma}} \int_0^1 |g_1 - g_2|^2 \, d\eta. \tag{2.10$'$}$$

If we use the choice of (2.8), inequalities (2.9) and $(2.9)'$, and the monotonicity of $\varphi_3(\cdot\,,\cdot)$, then we have

$$\begin{aligned} S_2 &= \int_0^1 \|A^{\beta+\gamma-1/2} w_2\|^{2\delta} \varphi_3^2[\|A^{\beta+\gamma-1/2} w_1\|, \|A^{\beta+\gamma-1/2} w_2\|] \\ &\quad \times [\|A^{\beta+\gamma} w_2\| + \|A^{\beta+\gamma} w_1\| \, \|A^{\beta+\gamma-1/2} w_2\|]^2 \, d\eta \\ &\le 2 \left[e^{2a\xi} \int_0^a |g_1 - g_2|^2 \, d\eta \right]^\delta e^{2\xi} \varphi_3^2(2C+2, 4\varepsilon_1^2) \left[(2C+2) \int_0^1 |g_1 - g_2|^2 \, d\eta \right]. \end{aligned} \tag{2.10$''$}$$

This, together with (2.8), implies the estimate

$$S_2 \le 8\varepsilon_1^{2\delta} e^{2\xi}(C+1)\varphi_3^2(2C+2, 4\varepsilon_1^2) \int\limits_0^1 |g_1 - g_2|^2\, d\eta.$$

The substitution of the estimates for S_1 and S_2 into (2.7)′ results in the inequality

$$\int\limits_0^1 \|F(g_1) - F(g_2)\|^2\, d\eta \le 8C_3^2(C+1)(1+C_1+C_2)^2 \times \left\{ \frac{\varphi_2^2(2C+2)}{(1+\xi)^{1-\gamma}} + \varepsilon_1^{2\delta} e^{2\xi} \varphi_3^2(2C+2, 4\varepsilon_1^2) \right\} \int\limits_0^1 |g_1 - g_2|^2\, d\eta, \qquad (2.11)$$

where C is the same as in (2.4), $\varphi_2(\cdot)$ and $\varphi_3(\cdot\,,\cdot)$ are the same as in Assumption B.1, and $C_3 = \sup_{0\le t\le 1} |\theta(t)|$, C_1, and C_2 are the same as in Lemma 2.2 for $\theta = \gamma$.

We choose ξ and ε_1 such that

$$\begin{gathered} [1600(C+1)^2(1+C_1+C_2)^2 C_3^2 \varphi_2^2(2C+2)]^{1/(1-\gamma)} \le \xi, \\ 0 \le \varepsilon_1 \le [1600(1+C_1+C_2)^2 C_3^2 e^{2\xi} \varphi_3^2(2C+2, 1)]^{-1/(2\delta)}. \end{gathered} \qquad (2.12)$$

Then inequality (2.11) acquires the form

$$\int\limits_0^1 \|F(g_1) - F(g_2)\|^2\, d\eta \le 0.01 \int\limits_0^1 |g_1 - g_2|^2\, d\eta. \qquad (2.11)'$$

Relation (2.6) with $g = 0$ becomes

$$F(0)(t) \equiv e^{-\xi t} A^\beta \theta(t)[B(\lambda, u) - B(\mu, u)] + e^{-\xi t} A^\beta \tilde{f}.$$

By using Assumption B.1, we obtain the inequality

$$\int\limits_0^1 \|F(0)(\eta)\|^2\, d\eta \le 2|\lambda - \mu|^2 \int\limits_0^1 e^{-\xi\eta} |\theta(t)|^2 \varphi_1^2(\|A^{\beta+\gamma-1/2} u\|_H) \|A^{\beta+\gamma} u\|_H^2\, d\eta + 2\int\limits_0^1 \|e^{-\xi\eta} A^\beta \tilde{f}\|^2\, d\eta.$$

This, together with condition (2.4) and Lemma 2.1 for $\theta = \beta + \gamma$, implies that

$$\int\limits_0^1 \|F(0)(\eta)\|^2\, d\eta \le 2CC_3^2 e^{-\xi t} |\lambda - \mu|^2 \varphi_1^2(C) + 2\int\limits_0^1 \|e^{-\xi\eta} A^\beta \tilde{f}\|^2\, d\eta, \qquad (2.11)''$$

where C is the same as in (2.4). Now, by assumption, the difference $\lambda - \mu$ is small enough to ensure that

$$2CC_3^2 e^{-2\xi} |\lambda - \mu|^2 \varphi_1^2(C) + 2\int\limits_0^1 \|e^{-\xi\eta} A^\beta \tilde{f}\|^2\, d\eta \le \frac{1}{16} \varepsilon_1^2 e^{-2\xi}, \qquad (2.11)'''$$

where ε_1 and ξ are the same as in Lemma 2.3.

Therefore, by virtue of (2.11)$'$, (2.11)$''$, (2.11)$'''$, and the choice (2.8) of the small ball M, one can apply the contraction mapping principle to Eq. (2.6) and prove the existence of g. Then

$$s(t) = \int_0^t e^{-(A+\xi)(t-\eta)}(A^{-\beta}g)(\eta)\,d\eta, \qquad w = e^{\xi t}s(t),$$

$$v = u - w = u - e^{\xi t}s(t) = u - e^{\xi t}\int_0^t e^{-(A+\xi)(t-\eta)}(A^{-\beta}g)(\eta)\,d\eta.$$

This, together with the fact that $g(\cdot)$ belongs to the small ball (2.8), readily implies the estimate (B.6)$'$.

To prove the estimate (B.6), we consider Eq, (2.1) itself, whence we obtain the inequality

$$\int_0^1 |A^\beta(u_t + Au)|^2\,dt \le 2\left(\int_0^1 |A^\beta f|^2\,dt + \int_0^1 |A^\beta B(\lambda,u)|^2\,dt\right).$$

Since $B(\lambda, 0) = 0$ (by virtue of Assumption B.1), it follows that the norm

$$\int_0^1 |A^\beta B(\lambda,u)|^2\,dt$$

can be estimated via $\sup_{0\le t\le 1}|A^{\beta+\gamma-1/2}u|(t)$ and $\int_0^1 |A^{\beta+\gamma}u(t)|^2\,dt$. By virtue of Lemma 2.1, these quantities can be estimated via the left-hand side of inequality (B.6)$'$. Therefore, by performing simple manipulations, we obtain the estimate (B.6). The proof of the lemma is complete.

Lemma 2.4. *Let*

$$\|A^\beta f\|^2_{H[0,1]} = \int_0^1 |A^\beta f|^2\,dt = \tilde{C}_0 < \infty \tag{2.13}$$

(β *is the same as in Assumption* B.1), *and let Assumptions* B.1 *and* B.2$'$ *be satisfied. Then the following assertions hold.*

(a) *If* $0 < \delta_1 < 1$, *then the solution* $u(\lambda, t)$ *of Eq.* (2.1) *satisfies the inequalities*

$$\sup_{|\lambda|<1-\delta_1}\int_0^1 |(A^\beta u_t(\lambda,t) + Au(\lambda,t))|^2\,dt \le M_{\delta_1} < \infty, \qquad \sup_{|\lambda|<1-\delta_1}\int_0^1 |A^\beta B(\lambda,u)|^2 dt < \infty,$$

where M_{δ_1} *depends on* δ_1 *and* $\tilde{C}_0$ *as well as on* α, β, *and* γ *in Assumptions* B.1 *and* B.2$'$ *and on values of the functions* $\varphi_1(\cdot)$, $\varphi_2(\cdot)$, $\varphi_3(\cdot,\cdot)$, *and* $\varphi_4(\cdot)$, *whose arguments are finite and depend on* δ, α, β, *and* γ.

(b) *If* $|\lambda| < 1$, *then, in the metric given by the norm*

$$\|A^\beta(u_t + Au)\|_{H[0,1]},$$

the solution $u(\lambda, t)$ *of problem* (2.1) *continuously depends on the function* f *changing in the metric with the norm* $\|A^\beta f\|_{H[0,1]}$. *Moreover, if*

$$\frac{du_j}{dt}(t) + Au_j(t) + B(\lambda, u_j) = f_j, \qquad u_j(0) = 0, \qquad j = 1, 2,$$

then

$$\left\| A^\beta \frac{d}{dt}(u_1 - u_2) + A(u_1 - u_2) \right\|_{H[0,1]} \le K \| A^\beta (f_1 - f_2) \|_{H[0,1]},$$

where K is a constant number depending on $\|u_1\|_{H[0,1]}$, $|\lambda| < 1$, α, β, and γ in Assumption B.2′ *and on values of the functions $\varphi_1(\cdot)$, $\varphi_2(\cdot)$, $\varphi_3(\cdot,\cdot)$, and $\varphi_4(\cdot)$, whose arguments are finite and depend on α, β, γ, and $\|u_1\|_{H[0,1]}$.*

Proof. Since, by the assumption of Lemma 2.4, Assumption $B.2'$ holds, it follows from condition (2.13) and condition (b) in Assumption $B.2'$ that inequality (2.4) holds for $\lambda = \lambda_0 \in \bar{G}$, where the constant C should be replaced by $C_1(\cdot)$ in $(2.2)'$.

Next, it follows from the assumption of Lemma 2.4 that the requirements of Lemma 2.3 are valid. Therefore, if $|\mu - \lambda_0| \le \varepsilon_0$, where ε_0 is the same number as in Lemma 2.3, then, by Lemma 2.3,

$$\int\limits_0^1 |A^\beta (v' + Av)|^2\, dt \le 8\varphi_4^2 (C+1)(C+1)^2, \qquad \int\limits_0^1 |A^{\beta+\gamma-1}(v' + Av)|^2\, dt \le 2C + 1, \tag{2.14}$$

where $C = C_1(\cdot)$ [$C_1(\cdot)$ is the same as in condition (b) in Assumption B.2′] and $\varphi_4(\cdot)$ is a continuous function from (B.3) on $[0, +\infty)$. For μ one can take the point $\tilde{\lambda}_0 = \lambda_0(1 - \varepsilon_0/2)$. Let $a(\cdot)$ be an analytic transformation of the unit disk into itself such that $a(\tilde{\lambda}_0) = 0$. Then $\partial\bar{G}$ is mapped into itself. By a classical theorem of the theory of analytic functions, there exists such a transformation. In this case, the image of the set $|\mu - \lambda_0| \le \varepsilon_0$, $\mu \in \bar{G}$, contains a disk neighborhood of zero, whose radius depends on ε_0 and is nonzero for $\varepsilon_0 = 0$. We denote this radius by ε_1.

Therefore, one can assume that the solution $u(\lambda, t)$ of Eq. (2.1) satisfies the first inequality in (2.14). This, together with Lemma 2.1, implies the inequality

$$\sup_{0 \le t \le 1} |A^{\beta+1/2} v|^2(t) + \int\limits_0^1 |A^{\beta+1} v|^2\, dt \le 2\tilde{C} \quad \text{if} \quad |\lambda| \le \varepsilon_1, \tag{2.15}$$

where $\tilde{C}$ is the right-hand side of the first inequality in (2.14). By Lemma 1.2, this inequality, together with the estimate (2.2), for $\alpha < \theta < 1 + \beta$ leads to the inequality

$$\sup_{|\lambda| \le r} \sup_{0 \le t \le 1} |A^{\theta - 1/2} v|(t) + \sup_{|\lambda| \le r} \left[\int\limits_0^1 |A^\theta v|^2\, dt \right]^{1/2} \le \frac{\tilde{C}_1}{1 - r\varepsilon_1^{-(\beta+1-\theta)/(\beta+1-\alpha)}}, \tag{2.16}$$

where $\tilde{C}_1 = 4(2C_0)^{2(\beta+1-\theta)/(\beta+1-\alpha)}(2\tilde{C})^{2(\theta-\alpha)/(\beta+1-\alpha)}$, $\tilde{C}$ and ε_1 are the same as in (2.15), and C_0 is the same as in condition (a) in Assumption B.2′; i.e.,

$$C_0^2 = \sup_{\lambda \in \partial G} \sup_{0 \le t \le 1} |A^{\alpha - 1/2} u(t)|^2 + \sup_{\lambda \in \partial G} \int\limits_0^1 |A^\alpha u|^2\, dt.$$

Set $\theta = \beta + \gamma$. Then one can take

$$r = \varepsilon_2 = \varepsilon_1^{1-\gamma}. \tag{2.17}$$

Now from (2.16) and the conditions of Assumption B.1, we have

$$\sup_{|\lambda| \le r} \int\limits_0^1 |A^\beta B(\lambda, v)|^2\, dt \le \tilde{C}_{1,1} < \infty, \tag{2.18}$$

where $\tilde{C}_{1,1}$ depends on $\tilde{C}$, C_0, and ε_1.

From (2.18) and Eq. (2.1), we obtain

$$\int_0^1 |A^\beta(v' + Av)|^2\, dt \le \tilde{C}_1 < \infty \quad \text{if} \quad |\lambda| \le \varepsilon_2, \tag{2.19}$$

where ε_2 is the same as in (2.17) and $\tilde{C}_1$ depends on $\tilde{C}$, C_0, and ε_1. We repeat these manipulations, replacing $\tilde{C}$ and ε_1 in (2.15) by $\tilde{C}_1$ and ε_2, respectively, taken from (2.19). Then

$$\int_0^1 |A^\beta(v' + Av)|^2\, dt \le \tilde{C}_2 < \infty \quad \text{if} \quad |\lambda| \le \varepsilon_3 = (\varepsilon_2)^{1-\gamma} = (\varepsilon_1)^{(1-\gamma)^2}.$$

By reproducing the argument n times, we obtain the inequality

$$\int_0^1 |A^\beta(v' + Av)|^2\, dt \le \tilde{C}_n < \infty \quad \text{if} \quad |\lambda| \le \varepsilon_n = (\varepsilon_1)^{(1-\gamma)^n}, \tag{2.19$'$}$$

where $\tilde{C}_n$ depends only on $\tilde{C}$, C_0, and ε_1. Since $0 < \gamma < 1$, it follows from (2.19)$'$ that the assertion of the lemma holds.

If A and $B(\lambda, \cdot)$ are continuous transformations and the solution of the Cauchy problem (B.2) exists for arbitrary $\lambda \in G = \{\lambda : \ |\lambda| < 1\}$ and $f(\lambda, \cdot) \in H_\beta[0, 1]$, then the validity of Theorem B.1 follows from Lemma 2.3 and assertion (b) of Lemma 2.4.

In the following section, we prove Theorems B.1 and B.2 in the general case. This possibility is provided by the fact that constants occurring in the estimates in Lemmas 2.3 and 2.4 depend only on constants occurring in the assumptions of the theorems.

3. PROOF OF THEOREM B.2

We use the notation of Theorem B.2 and Assumptions B.1, B.2, and B.2$'$.

Lemma 3.1. *If $\lambda, \mu \in \bar{G} = \{z : \ |z| \le 1\}$ and $u \in D(A^{\beta+\gamma})$, then*

$$\|A_N^\beta B_N(\lambda, u) - A_N^\beta B_N(\mu, u)\|_H \le |\lambda - \mu| \varphi_1(\|A_N^{\beta+\gamma-1/2} u\|_H) \|A_N^{\beta+\gamma} u\|_H.$$

Proof. We have

$$\begin{aligned}\|A_N^\beta B_N(\lambda, u) - A_N^\beta B_N(\mu, u)\|_H &\le \|A_N^\beta P_N(B(\lambda, P_N u) - B(\mu, P_N u))\|_H \\ &\le \|A_N^\beta (B(\lambda, P_N u) - B(\mu, P_N u))\|_H.\end{aligned}$$

This, together with condition (a) in Assumption B.1, implies that

$$\begin{aligned}\|A_N^\beta (B_N(\lambda, u) - B_N(\mu, u))\|_H &\le |\lambda - \mu| \varphi_1(\|A_N^{\beta+\gamma-1/2} P_N u\|_H) \|A_N^{\beta+\gamma} P_N u\|_H \\ &= |\lambda - \mu| \varphi_1(\|A_N^{\beta+\gamma-1/2} u\|_H) \|A_N^{\beta+\gamma} u\|_H.\end{aligned}$$

The proof of the lemma is complete.

In a similar way, one can derive the following assertion from condition (b) in Assumption B.1.

Lemma 3.2. *One has*

$$B_N(\lambda, u + w) - B_N(\lambda, u) = B_{0,N}(\lambda, u) w + B_{1,N}(\lambda, u, w),$$

where $B_{0,N}(\lambda, u) w = P_N B_0(\lambda, P_N u) P_N w$, $B_{1,N}(\lambda, u, w) = P_N B_1(\lambda, P_N u, P_N w)$, and $B_{0,N}(\lambda, \cdot)$ and $B_{1,N}(\lambda, \cdot, \cdot)$ are the same as in condition (b) *in Assumption* B.1.

Lemma 3.3. *The transformations* $B_{0,N}(\lambda,\cdot)$ *and* $B_{1,N}(\lambda,\cdot,\cdot)$ *introduced in Lemma* 3.2 *satisfy the estimates*

$$\|A_N^\beta B_{0,N}(\lambda,u)w\|_H \le [\|A_N^{\beta+\gamma}w\|_H + \|A_N^{\beta+\gamma-1/2}w\|_H\ |A_N^{\beta+\gamma}u\|_H]\varphi_2(\|A_N^{\beta+\gamma-1/2}u\|_H),$$
$$\|A_N^\beta B_{1,N}(\lambda,u,w)\|_H \le \varphi_3(\|A_N^{\beta+\gamma-1/2}u\|_H,\ \|A_N^{\beta+\gamma-1/2}w\|_H)\|A_N^{\beta+\gamma-1/2}w\|_H^\delta \times [\|A_N^{\beta+\gamma}w\|_H + \|A_N^{\beta+\gamma-1/2}w\|_H\|A_N^{\beta+\gamma}u\|_H].$$

The proof of this lemma is similar to that of Lemma 3.1.

Condition (B.3) implies the following assertion.

Lemma 3.4. *One has the relations* $B_N(\lambda,0)=0$ *and*

$$\|A_N^\beta B_N(\lambda,u)\|_H \le \|A_N^{\beta+\gamma}u\|_H\varphi_4(\|A_N^{\beta+\gamma-1/2}u\|_H).$$

By Lemmas 3.1, 3.2, 3.3, and 3.4 and Assumptions B.1 and B.2, from Lemma 2.4, we find that assertion (a) of Theorem B.2 holds.

Let $\varepsilon\in(0,1)$ and

$$u_N' + Au_N + B_N(\lambda,u_N) = P_Nf, \qquad u_N(0)=0, \qquad 0\le t\le 1.$$

Since $P_{N_1}P_{N_2}=P_{N_1}$, we have the relation

$$A^{\beta-\varepsilon}(u_{N_1}' - u_{N_2}' + A(u_{N_1}-u_{N_2})) = A^{\beta-\varepsilon}[B_{N_2}(\lambda,u_{N_2}) - B_{N_1}(\lambda,u_{N_1}) + (P_{N_1}-P_{N_2})f]$$
$$= (A^{\beta-\varepsilon}P_{N_2})[B(\lambda,u_{N_2}) - B(\lambda,u_{N_1}) + (P_{N_1}-P_{N_2})f]$$

for $1<N_1<N_2<\infty$, which, together with the inequality $|A^{-\varepsilon}P_{N_2}|\le N_2^{-\varepsilon}$, implies that

$$\|A^{\beta-\varepsilon}(u_{N_1}'-u_{N_2}'+A(u_{N_1}-u_{N_2}))\|_{H[0,1]} \le N_2^{-\varepsilon}\|A^\beta(B(\lambda,u_{N_2})-B(\lambda,u_{N_1}))\|_{H[0,1]} + N_2^{-\varepsilon}\|A^\beta f\|_{H[0,1]}.$$

By virtue of this inequality and Lemma 2.4, $A^{\beta-\varepsilon}(u_N'+Au_N)$ is a Cauchy sequence. Hence it follows that u_N has a weak limit u in the metric

$$(\|A^\beta u_N'\|^2_{H[0,1]} + \|A^{\beta+1}u_N\|^2_{H[0,1]})^{1/2}.$$

Therefore, $A^{\beta-\varepsilon}(u_N'+Au_N)$ converges to $A^{\beta-\varepsilon}(u'+Au)$. Then it follows from Lemma 2.1 that $A^{\beta+1-\varepsilon}u_N$ and $A^{\beta+1/2-\varepsilon}u_N$ are convergent in the metrics $\|\cdot\|_{H[0,1]}$ and $\sup_{0\le t\le 1}|\cdot|$, respectively. We choose ε from the relation $\beta+1-\varepsilon=\beta+\gamma$; i.e., $\varepsilon=1-\gamma$. Further, from condition (b) in Assumption B.1, for different N_1 and N_2, we obtain the relations

$$B(\lambda,u_{N_2}) - B(\lambda,u_{N_1}) = B_0(\lambda,u_{N_1})(u_{N_2}-u_{N_1}) + B_1(\lambda,u_{N_1},u_{N_2}-u_{N_1}),$$
$$\|A^\beta B_0(\lambda,u_{N_1})(u_{N_2}-u_{N_1})\|_H \le \varphi_2(\|A_N^{\beta+\gamma-1/2}u_{N_1}\|_H)[\|A^{\beta+\gamma}(u_{N_1}-u_{N_2})\|_H + \|A^{\beta+\gamma-1/2}(u_{N_1}-u_{N_2})\|_H\|A^{\beta+\gamma}u_{N_1}\|_H],$$
$$\|A^\beta B_1(\lambda,u_{N_1},(u_{N_1}-u_{N_2}))\|_H \le \|A^{\beta+\gamma-1/2}(u_{N_1}-u_{N_2})\|_H^\delta[\|A^{\beta+\gamma}(u_{N_1}-u_{N_2})\|_H + \|A^{\beta+\gamma-1/2}(u_{N_1}-u_{N_2})\|_H\|A^{\beta+\gamma}u_{N_1}\|_H] \times \varphi_3(\|A^{\beta+\gamma-1/2}u_{N_1}\|_H\|A^{\beta+\gamma-1/2}(u_{N_1}-u_{N_2})\|_H),$$

which imply that $A^\beta B(\lambda,u_N)$ is a Cauchy sequence in $H[0,1]$. Since the right-hand side of the equation

$$A^\beta\left[\frac{d}{dt}u_N + Au_N + B(\lambda,u_N)\right] = A^\beta[(E-P_N)B(\lambda,u_N)+P_Nf]$$

tends to zero in the metric of $H[0,1]$, it follows that $A^\beta\left[\frac{d}{dt}u_N + Au_N\right]$ is a Cauchy sequence in $[0,1]$ and is convergent. The limit of this sequence is $A^\beta\left[\frac{d}{dt}u + Au\right]$, since $A^{\beta-\varepsilon}\left[\frac{d}{dt}u_N + Au_N\right]$ converges to $A^{\beta-\varepsilon}\left[\frac{d}{dt}u + Au\right]$. Now it is obvious that u satisfies Eq. (2.1). The proof of assertion (b) of the theorem is complete. Assertion (c) of the theorem readily follows from the above-proved theorem and assertion (b) of Lemma 2.4. The proof of Theorem B.2 is complete.

To prove Theorem B.1, we apply Lemma 2.3 to the truncated systems (B.9) and pass to the limit as $N \to \infty$ (just as in the proof of Theorem B.2). The passage to the limit preserves the estimates. The proof of Theorem B.1 is complete.

4. WEAK A PRIORI ESTIMATES FOR SOLUTIONS OF A CLASS OF NONLINEAR EQUATIONS

Throughout this section, H is a Hilbert space of functions defined on some locally compact set. Let $A \ge E$ and $B(\cdot\,,\cdot)$ be a linear self-adjoint operator and a bilinear operator, respectively, in H taking real elements of H to real elements and having domains $D(A) \subset H$ and $D(B) \subset H$, respectively; i.e., A is defined on elements $x \in D(A)$, and $B(\cdot\,,\cdot)$ is defined on pairs (x, y) such that $x \in D(B)$ and $y \in D(B)$. We assume that $D(A) \subseteq D(B)$. This implies that the transformation B is subordinate to the operator A.

Consider system (B.11). We use the following condition on $B(\cdot\,,\cdot)$:

$$\langle B(g,u), v\rangle = -\langle B(g,v), u\rangle, \tag{4.1}$$

where $\langle\cdot\,,\cdot\rangle$ is the inner product in H.

Let

$$P_N u = \int_0^N dE_\lambda, \qquad B_N(u,v) = P_N B(P_N u, P_N v),$$

and let E_λ be the spectral resolution of identity corresponding to the self-adjoint operator A.

For $N \ge 1$, consider the Cauchy problem

$$\begin{gathered} u_k' + Au_k(t) + \alpha_{sj}^{(k)}\theta(t)B(u_s,u_j) = P_N f_k(t), \qquad k = 1,2, \\ u_1(t)|_{t=0} = u_2(t)|_{t=0} = 0, \qquad 0 \le t \le 1, \end{gathered} \tag{4.2}$$

where $f_1(t)$ and $f_2(t)$ are the same as in (B.11) and $u_1(t)$ and $u_2(t)$ are vector functions ranging in H. If A^{-1} is a compact operator, then system (4.2) is the Cauchy problem for a finite system of ordinary differential equations with real coefficients and with a real right-hand side.

Lemma 4.1. *If $g, u, v \in H$, then*

$$\langle B_N(g,u), v\rangle = -\langle B_N(g,v), u\rangle. \tag{4.1$'$}$$

Proof. By using the definition of B_N and the relations $P_N = P_N^*$ and $P_N^2 = P_N$, from condition (4.1), we obtain

$$\begin{aligned} \langle B_N(g,u), v\rangle &= \langle P_N B(P_N g, P_N u), P_N v\rangle = -\langle B(P_N g, P_N v), P_N u\rangle \\ &= -\langle P_N B(P_N g, P_N v), u\rangle = -\langle B_N(g,v), u\rangle. \end{aligned}$$

The proof of the lemma is complete.

Let $a = \{a_{kl}\}_{k,l=1}^2$ be a real-valued symmetric matrix. By multiplying the first equation in system (4.2) by $a_{11}u_1 + a_{12}u_2$ and the second one by $a_{12}u_1 + a_{22}u_2$ and by adding the resulting expressions, we obtain the relations

$$\langle u_k' + Au_k, a_{kl}u_l\rangle + \alpha_{sj}^{(k)}\theta(t)\langle B_N(u_s,u_j), a_{kl}u_l\rangle = \langle P_N f_k, a_{kl}u_l\rangle;$$

here the summation with respect to k, l, s, and j is implied. By taking into account (4.1)′ and the symmetry of the matrix a, we rewrite the last relations in the form

$$\left(a_{11}\frac{|u_1|^2}{2} + a_{22}\frac{|u_2|^2}{2} + a_{12}\langle u_1, u_2\rangle\right)'_t + a_{11}|\sqrt{A}u_1|^2 + a_{22}|\sqrt{A}u_2|^2$$
$$+ 2a_{12}\langle\sqrt{A}u_1, \sqrt{A}u_1\rangle + \theta(t)\langle B_N(u_1, u_1), u_2\rangle(\alpha_{11}^{(k)}a_{k2} - \alpha_{12}^{(l)}a_{l1})$$
$$+ \theta(t)\langle B_N(u_2, u_1), u_2\rangle(\alpha_{21}^{(k)}a_{k2} - \alpha_{22}^{(l)}a_{l1}) = a_{kl}\langle f_k, u_l\rangle. \tag{4.3}$$

We choose a_{11}, a_{22}, and a_{12} so as to satisfy the conditions

$$\alpha_{11}^{(k)}a_{k2} - \alpha_{12}^{(l)}a_{l1} = 0, \qquad \alpha_{21}^{(k)}a_{k2} - \alpha_{22}^{(l)}a_{l1} = 0, \qquad a_{12} \equiv a_{21}, \tag{4.4}$$

or, which is the same,

$$\begin{pmatrix} \alpha_{11}^{(2)} & -\alpha_{12}^{(1)} \\ \alpha_{21}^{(2)} & -\alpha_{22}^{(1)} \end{pmatrix} \begin{pmatrix} a_{22} \\ a_{11} \end{pmatrix} = a_{12} \begin{pmatrix} \alpha_{12}^{(2)} & -\alpha_{11}^{(1)} \\ \alpha_{22}^{(2)} & -\alpha_{21}^{(1)} \end{pmatrix}. \tag{4.4$'$}$$

Suppose that condition (B.12) is satisfied. Then from (4.4)′, we obtain

$$\begin{pmatrix} a_{22} \\ a_{11} \end{pmatrix} = \frac{a_{12}}{d} \begin{pmatrix} -\alpha_{22}^{(1)} & \alpha_{12}^{(1)} \\ -\alpha_{21}^{(2)} & \alpha_{11}^{(2)} \end{pmatrix} \begin{pmatrix} \alpha_{12}^{(2)} & -\alpha_{11}^{(1)} \\ \alpha_{22}^{(2)} & -\alpha_{21}^{(1)} \end{pmatrix}. \tag{4.5}$$

Lemma 4.2. *Let d be the same as in* (B.12), *let $d \neq 0$, let either $a_{12} \equiv a_{21} \equiv 1$ or $a_{12} \equiv a_{21} \equiv -1$, and let a_{11} and a_{22} be the numbers defined in* (4.5). *If $a_{11} > 0$ and $a_{11}a_{22} > 1$, then the solution of the Cauchy problem* (4.2) *satisfies the estimate* (B.13).

Proof. By virtue of (4.4), the terms in (4.3) containing $B_N(\cdot\,,\cdot)$ are zero. Therefore, by integrating relation (4.3) and by taking into account the initial conditions $u_1(0) = u_2(0) = 0$, we obtain

$$a_{11}\frac{|u_1|^2}{2} + a_{22}\frac{|u_2|^2}{2} + \langle u_1, u_2\rangle + \int_0^t (a_{11}|\sqrt{A}u_1|^2 + a_{22}|\sqrt{A}u_2|^2 + 2\langle\sqrt{A}u_1, \sqrt{A}u_2\rangle)\, d\eta$$
$$= \int_0^t (a_{11}\langle f_1, u_1\rangle + a_{22}\langle f_2, u_2\rangle + \langle f_1, u_2\rangle + \langle f_1, u_2\rangle)\, d\eta. \tag{4.6}$$

Let us estimate the right-hand side from above and the left-hand side from below. Since $a_{11}a_{22} > 1$, for the right-hand side of (4.6), we obtain the estimate

$$\int_0^t a_{kl}\langle f_k, u_l\rangle\, d\eta \le 2(a_{11} + a_{22}) \int_0^t \sqrt{f_1^2 + f_2^2}\, \sqrt{u_1^2 + u_2^2}\, d\eta$$
$$\le (a_{11} + a_{22}) \left[\frac{2}{\delta}\int_0^t (f_1^2 + f_2^2)\, d\eta + \frac{\delta}{2}\int_0^t (u_1^2 + u_2^2)\, d\eta\right]. \tag{4.7}$$

To estimate the left-hand side of (4.6) from below, we note that

$$a_{11}|u_1|^2 + a_{22}|u_2|^2 + 2\langle u_1, u_2\rangle = a_{11}|u_1 + a_{11}^{-1}u_2|^2 + (a_{22} - a_{11}^{-1})|u_2|^2 \ge (a_{22} - a_{11}^{-1})|u_2|^2.$$

In a similar way, we obtain the inequality

$$a_{11}|u_1|^2 + a_{22}|u_2|^2 + 2\langle u_1, u_2\rangle \ge (a_{11} - a_{22}^{-1})|u_1|^2.$$

Therefore,

$$a_{11}|u_1|^2 + a_{22}|u_2|^2 + 2\langle u_1, u_2\rangle \ge \frac{1}{2}\min\left(a_{11} - \frac{1}{a_{22}}, a_{22} - \frac{1}{a_{11}}\right)(|u_1|^2 + |u_2|^2).$$

In a similar way, we obtain the estimate

$$\begin{aligned} &a_{11}|\sqrt{A}u_1|^2 + a_{22}|\sqrt{A}u_2|^2 + 2\langle \sqrt{A}u_1, \sqrt{A}u_2\rangle \\ &\qquad \ge \frac{1}{2}\min\left(a_{11} - \frac{1}{a_{22}}, a_{22} - \frac{1}{a_{11}}\right)\left(|\sqrt{A}u_1|^2 + |\sqrt{A}u_2|^2\right). \end{aligned}$$

These two estimates provide a lower bound for the left-hand side of (4.6). Then, by using the estimate (4.7), we arrive at the inequality

$$\begin{aligned} &\frac{1}{4}\min\left(a_{11} - \frac{1}{a_{22}}, a_{22} - \frac{1}{a_{11}}\right)\left[|u_1|^2 + |u_2|^2 + \int_0^t (|\sqrt{A}u_1|^2 + |\sqrt{A}u_2|^2)\,d\eta\right] \\ &\qquad \le (a_{11} + a_{22})\left[\frac{2}{\delta}\int_0^t (f_1^2 + f_2^2)\,d\eta + \frac{\delta}{2}\int_0^t (u_1^2 + u_2^2)\,d\eta\right], \end{aligned}$$

where $\delta > 0$ is an arbitrary number. By setting

$$\delta = \frac{1}{4}\,\frac{1}{a_{11} + a_{22}}\min\left(a_{11} - \frac{1}{a_{22}}, a_{22} - \frac{1}{a_{11}}\right)$$

in this inequality and by using the relation $A \ge E$, we obtain

$$|u_1|^2 + |u_2|^2 + \int_0^t (|\sqrt{A}u_1|^2 + |\sqrt{A}u_2|^2)\,d\eta \le \frac{4}{\delta^2}\int_0^t (f_1^2 + f_2^2)\,d\eta,$$

which implies the assertion of the lemma.

Note that since $B_N(\cdot\,,\cdot)$ is a continuous mapping, it follows that system (4.2) has a solution on any time interval, and one can readily see that such a solution is unique if $B_N(\cdot\,,\cdot)$ satisfies the Lipschitz condition.

Proof of Theorem B.3. The a priori estimate of Theorem B.3 follows by the usual standard techniques from the estimate obtained in Lemma 4.2.

5. ON A STRONGLY SOLVABLE SYSTEM OF THE NAVIER–STOKES TYPE IN A HILBERT SPACE

We represent an analytic function $f(z)$ in G for $|z| = r < 1$ in the form

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} f_n r^n e^{in\varphi} = \sum_{n=0}^{\infty}(f_{n1} + if_{n2})(\cos n\varphi + \sin n\varphi)r^n \\ &= \sum_{n=0}^{\infty}(f_{n1}\cos n\varphi - f_{n2}\sin n\varphi)r^n + i\sum_{n=0}^{\infty}(f_{n1}\sin n\varphi + f_{n2}\cos n\varphi)r^n. \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{Re} f(z)|_{|z|=r} &= \sum_{n=0}^{\infty}(f_{n1}\cos n\varphi - f_{n2}\sin n\varphi)r^n, \\ \operatorname{Im} f(z)|_{|z|=r} &= \sum_{n=0}^{\infty}(f_{n1}\sin n\varphi + f_{n2}\cos n\varphi)r^n. \end{aligned} \tag{5.1}$$

We introduce an operator S acting on the trigonometric basis by the rule

$$S\sin n\varphi = \cos n\varphi, \qquad S\cos n\varphi = -\sin n\varphi \quad (n = 1, 2, \ldots), \qquad S\cdot 1 = 0. \tag{5.2}$$

From (5.1), we obtain the relation

$$\operatorname{Re} f(z)|_{|z|=r} = f_{01} + S\operatorname{Im} f(z)|_{|z|=r}. \tag{5.3}$$

Let $W_{2,\pi}^{(\alpha)}(-\pi,\pi)$ be the Sobolev space of periodic functions with the norm

$$\|m\|_{W_{2,\pi}^{(\alpha)}(-\pi,\pi)} = \left(\sum_{n=0}^{\infty}(n^2+1)^{\alpha}|m_{1n}|^2 + \sum_{n=0}^{\infty}(n^2+1)^{\alpha}|m_{2n}|^2\right)^{1/2},$$

where $m(\varphi) = \sum_{n=1}^{\infty} m_{1n}\sin n\varphi + \sum_{n=1}^{\infty} m_{2n}\cos n\varphi$.

Lemma 5.1. *Let $\alpha \ge 0$. The operator S is bounded from $W_{2,\pi}^{(\alpha)}(-\pi,\pi)$ to $W_{2,\pi}^{(\alpha)}(-\pi,\pi)$, and its norm is equal to unity. Moreover, if*

$$g(\varphi) = \sum_{n=1}^{\infty} g_{1n}\sin n\varphi + \sum_{n=1}^{\infty} g_{2n}\cos n\varphi,$$

then

$$\|Sg\|_{W_{2,\pi}^{(\alpha)}(-\pi,\pi)} = \|g\|_{W_{2,\pi}^{(\alpha)}(-\pi,\pi)}.$$

Proof. If

$$g(\varphi) = \sum_{n=1}^{\infty} g_{1n}\sin n\varphi + \sum_{n=1}^{\infty} g_{2n}\cos n\varphi - g_{21},$$

then

$$(Sg)(\varphi) = \sum_{n=1}^{\infty} g_{1n}\cos n\varphi - \sum_{n=1}^{\infty} g_{2n}\sin n\varphi.$$

Now let us compute the norms of g and Sg in the space $W_{2,\pi}^{(\alpha)}(-\pi,\pi)$:

$$\|Sg\|_{W_{2,\pi}^{(\alpha)}(-\pi,\pi)} = \sum_{n=1}^{\infty}(n^2+1)^{\alpha}|g_{1n}|^2 + \sum_{n=1}^{\infty}(n^2+1)^{\alpha}|g_{2n}|^2 = \|g\|_{W_{2,\pi}^{(\alpha)}(-\pi,\pi)} - |g_{21}|^2.$$

This implies the assertion of the lemma.

Lemma 5.2. *If $f, g \in W_{2,\pi}^{(1)}(-\pi,\pi)$, then $fg \in W_{2,\pi}^{(1)}(-\pi,\pi)$; moreover,*

$$\|fg\|_{W_{2,\pi}^{(1)}(-\pi,\pi)} \le C\|f\|_{W_{2,\pi}^{(1)}(-\pi,\pi)}\|g\|_{W_{2,\pi}^{(1)}(-\pi,\pi)}.$$

Proof. The norm in $W_{2,\pi}^{(1)}(-\pi,\pi)$ is equivalent to the norm

$$\|f\|^2_{W_{2,\pi}^{(1)}(-\pi,\pi)} = \|f'\|^2_{L_2(-\pi,\pi)} + \|f\|^2_{L_2(-\pi,\pi)}.$$

Consequently,

$$\begin{aligned}
\|fg\|_{W_{2,\pi}^{(1)}(-\pi,\pi)} &\le \left\|\frac{d}{dt}(fg)\right\|_{L_2(-\pi,\pi)} + \|fg\|_{L_2(-\pi,\pi)} \\
&\le C_0\Big[\|f'g\|_{L_2(-\pi,\pi)} + \|fg'\|_{L_2(-\pi,\pi)} + \|g\|_{L_2(-\pi,\pi)}\|f\|_{C[-\pi,\pi]}\Big] \\
&\le C_1\Big[\|g\|_{C[-\pi,\pi]} + \|f\|_{C[-\pi,\pi]}\Big]\Big[\|f'\|_{L_2(-\pi,\pi)} + \|g'\|_{L_2(-\pi,\pi)} \\
&\quad + \|g\|_{L_2(-\pi,\pi)} + \|f\|_{L_2(-\pi,\pi)}\Big].
\end{aligned}$$

Note that

$$\|f\|_{C[-\pi,\pi]} \le C\Big[\|f'\|_{L_2(-\pi,\pi)} + \|f\|_{L_2(-\pi,\pi)}\Big].$$

Then it follows from the preceding inequality that

$$\|fg\|_{W^{(1)}_{2,\pi}(-\pi,\pi)} \le C_2\|f\|_{W^{(1)}_{2,\pi}(-\pi,\pi)}\|g\|_{W^{(1)}_{2,\pi}(-\pi,\pi)}.$$

The proof of the lemma is complete.

In (B.15), we take $b_{sj} = \delta_{sj} + \varepsilon\tilde b_{sj}$ $(s, j = 1, 2)$; i.e.,

$$b = \operatorname{diag}(1,1) + \varepsilon(\tilde b_{sj})^2_{s,j=1}, \tag{5.4}$$

where the $\tilde b_{sj}$ are complex-valued functions defined on ∂D, ε is a small number, and δ_{sj} is the Kronecker delta.

For the matrix $\gamma = \{\gamma_{sj}\}$, we have $\gamma_{sj} = \delta_{sj} - \varepsilon\tilde b_{sj} + \varepsilon^2 o(1)$; i.e.,

$$\gamma = \operatorname{diag}(1,1) - \varepsilon(\tilde b_{sj} + \varepsilon o(1))^2_{s,j=1}. \tag{5.5}$$

By virtue of (5.4) and (5.5), for the coefficients of Eq. (B.11)$'$, we obtain the chain of inequalities

$$\begin{aligned} d^{(l)}_{mn} &\equiv \alpha^{(k)}_{sj} b_{lk}\gamma_{sm}\gamma_{jn} = \alpha^{(k)}_{sj}(\delta_{lk} + \varepsilon\tilde b_{lk})(\delta_{sm} - \varepsilon\tilde b_{sm})(\delta_{jn} - \varepsilon\tilde b_{jn}) + \varepsilon^2 o(1) \\ &= \alpha^{(l)}_{mn} + \varepsilon[\alpha^{(k)}_{sj}\tilde b_{lk}\delta_{sm}\delta_{jn} - \alpha^{(k)}_{sj}\tilde b_{sm}\delta_{lk}\delta_{jn} - \alpha^{(k)}_{sj}\tilde b_{jn}\delta_{sm}\delta_{lk}] + \varepsilon^2 o(1) \\ &= \alpha^{(l)}_{mn} + \varepsilon[\alpha^{(k)}_{mn}\tilde b_{lk} - \alpha^{(l)}_{sn}\tilde b_{sm} - \alpha^{(l)}_{mj}\tilde b_{jn}] + \varepsilon^2 o(1). \end{aligned} \tag{5.6}$$

Theorem 5.1. *Let $C^{(l)}_{mn}$ $(l, m, n = 1, 2)$ be real numbers, and let $\tilde b_{lk}$ $(l, k = 1, 2)$ be arbitrary complex-valued functions defined on ∂G and belonging to the Sobolev class $W^{(1)}_2(\partial G)$. Then there exist functions $\alpha^{(l)}_{sj}$ $(l, s, j = 1, 2)$ and g_k $(k = 1, 2)$ defined on ∂G, belonging to the class $W^{(1)}_2(\partial G)$, and such that the functions $\alpha^{(k)}_{sj} b_{lk}\gamma_{sm}\gamma_{jn} \equiv d^{(l)}_{mn}$ $(l, m, n = 1, 2)$ and $b_{lk}g_k \equiv \theta_j$ $(k = 1, 2)$ are the values on ∂G of analytic functions $d^{(l)}_{mn}(z)$ and $\theta_j(z)$ $(l, m, n = 1, 2)$ in G.*

Note that $W^{(1)}_2(\partial G)$ is just $W^{(1)}_{2,\pi}(-\pi,\pi)$ if the functions defined on ∂G are identified with the periodic functions defined on $(-\pi,\pi)$. This is possible by virtue of the relation

$$\partial G = \{z: \ |z| = 1\} = \{z: \ z = e^{i\varphi}, \ -\pi < \varphi < \pi\}.$$

Proof of Theorem 5.1. We choose $\alpha^{(l)}_{mn} = \alpha^{(l)}_{mn}(\varphi)$ on ∂G from the equations

$$\begin{aligned} &\alpha^{(l)}_{mn} + \operatorname{Re}\{\varepsilon[\alpha^{(k)}_{mn}\tilde b_{lk} - \alpha^{(l)}_{sn}\tilde b_{sm} - \alpha^{(l)}_{mj}\tilde b_{jn}] + \varepsilon^2 o(1)\} \\ &\quad = C^{(l)}_{mn} + S\operatorname{Im}\{\varepsilon[\alpha^{(k)}_{mn}\tilde b_{lk} - \alpha^{(l)}_{sn}\tilde b_{sm} - \alpha^{(l)}_{mj}\tilde b_{jn}] + \varepsilon^2 o(1)\}, \end{aligned} \tag{5.7}$$

or

$$\operatorname{Re} d^{(l)}_{mn} = S\operatorname{Im} d^{(l)}_{mn} + C^{(l)}_{mn},$$

where $d^{(l)}_{mn}$ are the same as in (5.6), i.e., from a system of eight linear equations for the real functions $\alpha^{(l)}_{mn}$ $(l, m, n = 1, 2)$. By setting $L = \{\alpha^{(l)}_{mn}\}$ and $C = C^{(l)}_{mn}$, for small real ε, one can rewrite this system in the form

$$L = C + K(\varepsilon)L, \tag{5.7$'$}$$

where, by Lemmas 5.1 and 5.2, $K(\varepsilon)$ is an operator small in norm. Consequently, Eq. (5.7)$'$ [and hence (5.7)] has a solution.

Now for $d_{mn}^{(l)}$, we have the relations

$$\begin{aligned}
d_{mn}^{(l)} &= \operatorname{Re} d_{mn}^{(l)} + i \operatorname{Im} d_{mn}^{(l)} = i \operatorname{Im} d_{mn}^{(l)} + S \operatorname{Im} d_{mn}^{(l)} + C_{mn}^{(l)} \\
&= C_{mn}^{(l)} + S\left(\sum_{k=1}^{\infty} \gamma_{mn}^{(l)} \sin kx + \sum_{k=1}^{\infty} \gamma_{mn}^{(l)} \cos kx\right) + i\left(\sum_{k=1}^{\infty} \gamma_{mn}^{(l)} \sin kx + \sum_{k=1}^{\infty} \gamma_{mn}^{(l)} \cos kx\right) \\
&= C_{mn}^{(l)} + i(\gamma_{mn}^{(l)})_{02} + \sum_{k=1}^{\infty}\{(\gamma_{mn}^{(l)})_{k1}(\cos kx + i \sin kx) + (\gamma_{mn}^{(l)})_{k2}(-\sin kx + i\cos kx)\} \\
&= C_{mn}^{(l)} + i(\gamma_{mn}^{(l)})_{02} + \sum_{k=1}^{\infty}(\gamma_{mn}^{(l)})_{k1} e^{ikx} + i\sum_{k=1}^{\infty}(\gamma_{mn}^{(l)})_{k2} e^{ikx} \\
&= C_{mn}^{(l)} + i(\gamma_{mn}^{(l)})_{02} + \sum_{k=1}^{\infty}[(\gamma_{mn}^{(l)})_{k1} + i(\gamma_{mn}^{(l)})_{k2}] e^{ikx}
\end{aligned}$$

on ∂G. Here $(\gamma_{mn}^{(l)})_{k1}$ and $(\gamma_{mn}^{(l)})_{k2}$ are the Fourier coefficients of the function $\operatorname{Im} d_{mn}^{(l)}$. By virtue of Eqs. (5.7) and (5.7)$'$, Lemmas 5.1 and 5.2, and the condition $\tilde{b}_{lk} \in W_2^{(1)}(\partial G)$, the function $\operatorname{Im} d_{mn}^{(l)}$ belongs to the class $W_2^{(1)}(\partial G)$ on ∂G. Therefore, it follows from the last relation that $d_{mn}^{(l)}$ is the value on ∂G of the analytic function

$$d_{mn}^{(l)}(z) = C_{mn}^{(l)} + i(\gamma_{mn}^{(l)})_{02} + \sum_{k=1}^{\infty}\Big[(\gamma_{mn}^{(l)})_{k1} + i(\gamma_{mn}^{(l)})_{k2}\Big] z^k$$

in G.

To prove the assertions of the theorem about the functions $\theta_j(z)$, let us write out conditions under which the $b_{lk}g_k$ are the values of analytic functions. Arguing as in the proof of the assertions of the theorem on the functions $d_{mn}^{(l)}(z)$, we find that g_1 and g_2 can be chosen so as to ensure that $b_{lk}g_k$ $(l = 1, 2)$ are the values on ∂G of the analytic function $\theta_j(z)$ in G. The proof of the theorem is complete.

Theorem 5.2. *Let $d_{mn}^{(l)}(z)$ and $\theta_j(z)\alpha_{mn}^{(l)}$ $(l, m, n = 1, 2)$ be the analytic functions in*

$$G = \{z : \ |z| < 1\}$$

constructed in Theorem 5.1, *where the $\tilde{b}_{sj}$ are defined on ∂G in accordance with* (5.4), *and let ε be a small number. Suppose that the Cauchy problem for the system of equations*

$$\begin{gathered}
\frac{d}{dt}v_1(t) + Av_1(t) + d_{mn}^{(l)}(z)B(v_m, v_n) = \theta_1(t)f(t), \\
\frac{d}{dt}v_2(t) + Av_2(t) + d_{mn}^{(l)}(z)B(v_m, v_n) = \theta_2(t)f(t), \\
v_1(t)|_{t=0} = v_2(t)|_{t=0} = 0, \qquad 0 \le t \le 1,
\end{gathered} \tag{5.8}$$

with some $z \in \bar{G}$ has a strong solution. Then problem (5.8) *has a strong solution for all $z \in G$, $z < 1$.*

Proof. By using the transformation inverse to (5.4), one can pass from system (5.8) with $z \in \partial G$ to system (B.11). The solution of system (B.11) satisfies the a priori estimates in Theorem B.3. By using the transformation (5.4), for $z \in \partial G$, we pass from system (B.11) to system (5.8) and obtain weak a priori estimates for the solution of system (5.8) for $z \in \partial G$.

Now one can readily see that all assumptions of Theorem B.2 hold. Therefore, the assertion of Theorem 5.2 follows from Theorem B.2.

6. REDUCTION OF THE NAVIER–STOKES EQUATIONS TO AN ABSTRACT EQUATION IN A HILBERT SPACE

Consider the system of Navier–Stokes equations

$$\frac{\partial}{\partial t}u_j - \Delta u_j + u_k\partial_k u_j + \partial_j p = f_j, \qquad \operatorname{div} u \equiv \partial_j u_j = 0, \tag{6.1}$$

where $t \in (0,\infty)$, $x = (x_1,\dots,x_n) \in R^n$, $\partial_k = \dfrac{\partial}{\partial x_k}$, $\Delta = \sum_{k=1}^n \partial_k^2 = \partial_k\partial_k$ is the Laplace operator, $u = (u_1,\dots,u_n)$ and $f = (f_1,\dots,f_n)$ are vector functions of the arguments $t \in (0,+\infty)$ and $x \in R^n$, and $p(t,x)$ is a scalar function (the pressure). We supplement system (6.1) with the Cauchy initial conditions

$$u_j(t)|_{t=0} = 0 \qquad (j = 1,2,\dots,k). \tag{6.2}$$

By using standard techniques, from system (6.1), we obtain the estimate

$$\frac{1}{2}\|u(t,\cdot)\|^2_{L_2(R^n)} + \int_0^t \|\nabla u\|^2_{L_2(R^n)}\,d\eta \le 2C\int_0^t |f|^2\,d\eta. \tag{6.3}$$

The a priori estimate (6.3) permits one to prove the existence of a weak Hopf solution [2, p. 83 of the Russian translation].

Throughout the following, it is convenient to set $u_j = e^t v_j$ and rewrite Eq. (6.1) in the form

$$\partial_k v_j + (-\Delta + E)v_j + e^t v_k\partial_k v_j + e^{-t}\partial_j p = e^{-t}f_j, \qquad \operatorname{div} v = 0, \qquad v_j(t)|_{t=0} = 0 \tag{6.1$'$}$$

or, as usual,

$$\partial_k v + (-\Delta + E)v + e^t(v,\nabla)v + e^{-t}\operatorname{grad} p = e^{-t}f, \qquad \operatorname{div} v = 0, \qquad v(t)|_{t=0} = 0. \tag{6.1$''$}$$

Now instead of the estimate (6.3), one can obtain the estimate

$$\|v(t)\|^2_{L_2(R^n)} + \int_0^t \|\nabla v\|^2_{L_2(R^n)}\,d\eta + \int_0^t \|v\|^2_{L_2(R^n)}\,d\eta \le \int_0^t \|(-\Delta + E)^{-1/2}f(\cdot)\|^2_{L_2(R^n)}\,d\eta. \tag{6.3$'$}$$

The estimates (6.3) and (6.3)$'$ for $n \ge 3$ are insufficient for

(a) proving the uniqueness of the solution;

(b) using perturbation theory;

(c) increasing the smoothness of the solution together with the smoothness of the right-hand side.

The issues mentioned in (a), (b), and (c) (or at least one of them) are important in applications to problems of mechanics and gas (magnetic or nonmagnetic) dynamics; in particular, they would play an important role in the proof of the passage of a laminar flow of a fluid to a turbulent one. Problems (a), (b), and (c) are directly related to the problem of the strong solvability in the whole for problem (6.1). Numerous papers deal with attempts to solve the problem of the strong solvability for problem (6.1). The above-proved theorems B.1–B.3 are also corollaries of attempts of one of the authors.

We introduce the operator R acting by the formula

$$\mathrm{R}v = (E - \operatorname{grad}\Delta^{-1}\operatorname{div})v = v - \operatorname{grad}\Delta^{-1}\operatorname{div} v.$$

This operator acts (and is bounded) in all Sobolev spaces $W_2^{(\alpha)}(R^n)$ [$\alpha \in (-\infty,\infty)$] and is an orthogonal projection.

By applying the operator R to system (6.1)$''$, we obtain the problem

$$\partial_k v + (-\Delta + E)v + e^t\mathrm{R}(v,\nabla)v = e^{-t}\mathrm{R}f, \qquad \operatorname{div} v = 0, \qquad v(t)|_{t=0} = 0. \tag{6.4}$$

We have eliminated the pressure p. In the derivation of (6.4) from (6.1)$''$, we have used the fact that

$$\text{if} \quad \operatorname{div} v = 0, \quad \text{then} \quad \mathrm{R}v = v. \tag{6.5}$$

By taking into account condition (6.5), from (6.4), we obtain the problem

$$\partial_k v + (-\Delta + E)v + e^t \mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}v = e^{-t}\mathrm{R}f, \qquad \operatorname{div} v = 0, \qquad v(t)|_{t=0} = 0. \qquad (6.4)'$$

Instead of (6.4)$'$, we consider the equation

$$\partial_k v + (-\Delta + E)v + e^t \mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}v = g, \qquad v(t)|_{t=0} = 0, \qquad 0 < t < \infty. \qquad (6.4)''$$

System (6.4)$''$ does not contain the equation $\operatorname{div} v = 0$. But if $\operatorname{div} g = 0$, then, by applying the operator R to (6.4)$''$, we obtain the equation

$$\partial_t \operatorname{div} v + (-\Delta + E)\operatorname{div} v = 0, \qquad \operatorname{div} v(t)|_{t=0} = 0.$$

Hence it follows that $\operatorname{div} v = 0$. Therefore, instead of (6.4)$'$ [or (6.4) and (6.1)], one can consider (6.4)$''$ provided that the function g satisfies the condition $\operatorname{div} g = 0$.

By denoting the space $L_2(R^n)$ by H, the operator $(-\Delta+E)$ by A, and the operator $\mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}v$ by $B(\cdot\,,\cdot)$, one can rewrite Eq. (6.4)$''$ as the abstract equation

$$u' + Au + B(u,u) = f(t), \qquad u(0) = 0. \tag{6.6}$$

One can readily see that

$$\langle B(u,g), w\rangle = -\langle B(u,w), g\rangle;$$

i.e., condition (4.1) is satisfied.

In the spaces $L_p(R^n)$ $(1 < p < \infty)$, the fractional powers of the operator $(-\Delta + E)$ can be defined by the relations

$$(-\Delta + E)^\alpha v(x) = F^{-1}_{\xi\to x}(|\xi|^2 + 1)^\alpha F_{y\to\xi} v(y),$$

where $\alpha \in (-\infty, +\infty)$. This operator is known to satisfy the estimates

$$\|(-\Delta + E)^\beta v\|_q \le C\|(-\Delta + E)^\alpha v\|_p, \tag{6.7}$$

valid for $1 < p \le q < \infty$ and $2(\alpha - \beta) \ge n(1/p - 1/q)$. These inequalities are known as exact embedding theorems and were obtained in [5].

If $1 < p \le 2$ and $-2\beta = n(1/p - 1/2)$, then, by using (6.7), we obtain the chain of inequalities

$$\begin{aligned}\|(-\Delta + E)^\beta \mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}u\|_{L_2(R^n)} &\le \|(-\Delta + E)^\beta (\mathrm{R}v, \nabla)\mathrm{R}u\|_{L_2(R^n)} \\ &\le C\|(\mathrm{R}v, \nabla)\mathrm{R}u\|_{L_2(R^n)} \le C_0\|\mathrm{R}v\|_{L_{2p}(R^n)}\|\operatorname{grad}\mathrm{R}u\|_{L_{2p}(R^n)} \\ &\le C_1\|\mathrm{R}v\|_{L_{2p}(R^n)}\|(-\Delta + E)^{-1/2}u\|_{L_{2p}(R^n)},\end{aligned}$$

which, together with inequality (6.7), implies the estimate

$$\|(-\Delta + E)^\beta \mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}u\|_{L_2(R^n)} \le C_2\|(-\Delta + E)^\alpha \mathrm{R}v\|_{L_2(R^n)}\|(-\Delta + E)^{\alpha+1/2}\mathrm{R}u\|_{L_2(R^n)}, \tag{6.8}$$

where $1 < p \le 2$ and $2\alpha \ge n(1/2 - 1/(2p)) = n/4 + (n/2)(1/2 - 1/p) = n/4 + \beta$. The estimate (6.8) provides the condition for the subordination of the nonlinear part of problem (6.4)$''$ to the linear one for $\beta \in (-n/4, 0]$. Inequality (6.8) with $\beta \ge 0$ holds for $2\alpha \ge k + n/4 + \tilde\beta$, k is an integer, and $\tilde\beta \in (-n/4, 0]$. Indeed, for $\beta > 0$, we have

$$\begin{aligned}|(-\Delta + E)^\beta \mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}u| &= |(-\Delta + E)^{\tilde\beta + k/2}\mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}u| \\ &\le C\sum_{j=1}^n |(-\Delta + E)^{\tilde\beta} D^\alpha \mathrm{R}(\mathrm{R}v, \nabla)\mathrm{R}u|.\end{aligned} \tag{6.9}$$

Here

$$D^\alpha = \prod_{j=1}^{n} \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j},$$

and the sum extends over all n-dimensional vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with real nonnegative components such that $0 \le \sum_{j=1}^{n} \alpha_j \le k$. Now inequality (6.8) can be derived from (6.9) just as in the case $\beta \in (-n/4, 0]$ with the use of embedding theorems. Note that inequality (6.8) was obtained in [12, 13] only for $\beta \in (-n/4, 0]$. In [12], the number γ occurring in conditions (2) and (3) was chosen in the interval $(0, 3/4)$ rather than $(-\infty, 3/4)$.

Note that the present research is close to the papers [6–13].

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