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Existence Conditions for a Global Strong Solution to One Class of Nonlinear Evolution Equations in a Hilbert Space

M. Otelbaev, A. A. Durmagambetov, and E. N. Seitkulov

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Let H be a separable real Hilbert space. The norm in H is denoted by $|\cdot|$, and the inner product, by $\langle\cdot, \cdot\rangle$. Let $C^\infty(H; 0, a)$, where $a > 0$, be the set of infinitely smooth functions on $[0, a]$ with values in H. The completion of $C^\infty(H; 0, a)$ with respect to the metric defined by the inner product

$$\langle x, y\rangle_{H_2} = \int_0^a \langle x(t), y(t)\rangle dt, \quad x(t), y(t) \in C^\infty(H; 0, a)$$

is denote as $H_2 = H_2[0, a]$.

Let A be a self-adjoint nonnegative operator with a completely continuous inverse, and let $D(A)$ denote the domain of A. In the space of functions with values in H, we consider the Cauchy problem

$$u'_t + Au + B(u, u) = f(t), \quad u(0) = 0, \quad 0 < t < a, \tag{1}$$

where $B(u, g)$ is a bilinear operator and $f(t)$ is a function with values in H.

The system of Navier–Stokes equations can be written in the form of (1), and the well-known problem of the existence of a global strong solution is reduced to a similar problem for abstract equation (1).

In this paper, we consider the global strong solvability of problem (1), assuming that A and $B(\cdot, \cdot)$ are related so that $B(\cdot, \cdot)$ is subordinated to A. More specifically, it is assumed that

$$|(A+E)^{-\gamma}B(u, g)| \le c\Big[|(A+E)^{\gamma_0}u|\,|(A+E)^{\gamma_0+\frac{1}{2}}g| + |(A+E)^{\gamma_0}g|\,|(A+E)^{\gamma_0+\frac{1}{2}}u|\Big] \tag{2}$$

for all γ and γ_0 that satisfy the relations

$$\gamma_0 = \delta_0 - \frac{\gamma}{2}, \quad -\infty < \gamma \le \frac{3}{4}, \quad 0 < \delta_0 < \frac{1}{2}, \tag{3}$$

where δ_0 is a constant and c depends on γ. Here, (2) and (3) are the conditions for the subordination of the nonlinear operator $B(\cdot, \cdot)$ to A. For the three-dimensional Navier–Stokes equations, these conditions are fulfilled at $\delta_0 = \frac{3}{8}$. Subordination conditions (2) and (3) were picked according to the inequalities satisfied by the nonlinear term in the Navier–Stokes equations. In fact, these conditions can be chosen in a different manner. For example, we can require that condition (3) hold only for some γ rather than for all $\gamma \in \left(-\infty, \frac{3}{4}\right]$.

Definition 1. Problem (1) is said to be globally strongly solvable if, for any $a > 0$, the condition $f(t) \in H_2[0, a]$ implies that problem (1) has a solution $u(t)$ in $(0, a)$ such that $u' + Au \in H_2[0, a]$.

Below is the main result of this work.

Theorem A. *Let* $A \ge E$, *and let conditions* (2) *and* (3) *be satisfied. Problem* (1) *is globally strongly solvable if and only if the system of equations*

$$\begin{gathered} -s'(t) + As(t) + B_w^* s(t) = 0, \quad s(a) = 0, \quad 0 < t < a, \\ w' + Aw(t) + B(w, w) = s(t), \\ w(0) = 0, \quad 0 < t < a \end{gathered} \tag{4}$$

has only the trivial solution $(w(t), s(t))$; *i.e.,* $s(t) \equiv 0$ *and* $w(t) \equiv 0$.

Here, B_w^* is the adjoint of B_w and B_w is defined by the formula $B_w g = B(w, g) + B(g, w)$. Below, we explain the meaning of the word "solution" used in this theorem.

Definition 2. The test functions, denoted by D_a, are the set of all pairs $\{w_1, w_2\}$ of infinitely smooth vector

Gumilev Eurasian National University, Astana, Republic of Kazakhstan

functions with values in $D(A) \subseteq H$ that obey the following conditions:

(a) $w_1 = 0$ in the neighborhood of $t = 0$, and $w_2 = 0$ in the neighborhood of $t = a$.

(b) $Aw_j \in H_2[0, a]$, $j = 1, 2$.

Definition 3. A pair of vector functions $(w(t), s(t))$ is called a solution to problem (4) if $s(t) \in H_2[0, a]$ and the following conditions are fulfilled:

(c) For any $\delta \in (0, a)$, it is true that $w' + Aw \in H_2[0, a - \delta]$.

(d) For any pair $(w_1, w_2) \in D_a$,

$$\langle s, w_1' + Aw_1 + B_w w_1 \rangle_{H_2} = 0,$$

$$\left\langle -w_2' + Aw_2 + \frac{1}{2}B_w^* w_2, w \right\rangle_{H_2} - \langle w_2, s \rangle_{H_2} = 0.$$

Here and below, $H_2 = H_2[0, a]$. Thus, Theorem A deals with a "weak" solution.

Note that the equation for $s(t)$ is "inversely parabolic" and the Cauchy condition is set at its right end. Moreover, this is a linear homogeneous equation. Therefore, if $w(t)$ in the neighborhood of $t = a$ is such that $B_w^* s(t)$ is subordinated to $-s'(t) + As(t)$, then the problem has only the trivial solution. However, since the vector function $w(t)$ may have a singularity at $t = a$, the problem for $s(t)$ possibly has a nontrivial solution. Therefore, Theorem A does not solve the problem of the existence of a global strong solution but reduces it to another problem.

Eliminating $s(t)$ from system (4), we obtain the following two-point problem for $w(t)$:

$$\left(-\frac{d^2}{dt^2} + A^2\right)w + \left(-\frac{d}{dt} + A\right)B(w, w)$$

$$+ B_w^*\left(\frac{d}{dt}w + Aw + B(w, w)\right) = 0,$$

$$(w' + Aw(t) + B(w, w))|_{t = a} = 0, \quad w(0) = 0.$$

Although the linear part (which seems to be the principal part) in this problem is "elliptic," we fail to prove that its solution is trivial. For the Navier–Stokes equations, the weak (energy) Hopf estimate holds. We do not use such an estimate, and our main result holds true without assuming weak a priori estimates. That is why our main result will remain valuable even if the existence of a global strong solution to the Navier–Stokes equations is proved.

Definition 4. A vector function $f(t) \in H_2[0, a]$ with $|f|_{H_2} \neq 0$ is called a separating function for problem (1) if the following conditions are satisfied:

(a) The solution $u(t)$ to problem (1) in $(0, a)$ exists and $u' + Au \in H_2[0, a - \varepsilon]$ for any $\varepsilon \in (0, a)$, but $u' + Au \notin H_2[0, a]$.

(b) If $g(t) \in H_2[0, a]$ and $|g|_{H_2} < |f|_{H_2}$, then the solution to the problem

$$v'(t) + Av + B(v, v) = g(t), \quad v(0) = 0,$$
$$0 < t < a$$

exists and $v'(t) + Av \in H_2[0, a]$.

The proof of the theorem is based on the following two lemmas.

Lemma 1. *If there exists $f(\cdot) \in H_2[0, a]$ such that problem* (1) *has no solution satisfying $u' + Au \in H_2[0, a]$, then problem* (1) *has a separating function.*

Lemma 2. *Let $s(t)$ be a separating function, $|s|_{H_2[0, a]} > 0$, and $w(t)$ be the solution to problem* (1) *with $f = s(t)$. Then the pair (w, s) satisfies system* (4).

The following result can be proved.

Theorem B. *Let $f(t)$ be a separating function for problem* (1) *and $u(t)$ be the corresponding solution to problem* (1) *in $(0, a)$. Then $(|u_t'|^2 - |Au + B(u, u)|^2)_t' = 0$ for $t \in (0, a)$.*

Denoting $u' + Au$ by v, we can rewrite problem (1) as

$$v + B(Tv, Tv) = f, \quad Tv = \int_0^t e^{-A(t - \eta)} v(\eta) d\eta = u(t). \tag{5}$$

Let $H_{p, \theta}(0, a)$ be the space of H-valued vector functions that is the completion of $C(H; 0, a)$ $(a > 0)$ with respect to the norm

$$\|f(\cdot)\| = \left(\int_0^a |A^\theta f(\eta)|_H^p d\eta\right)^{\frac{1}{p}}.$$

When the value of $a > 0$ does not need to be traced, we replace $H_{p, \theta}(0, a)$ with $H_{p, \theta}$. Below are the definitions that are necessary in what follows.

Definition 5. Equation (5) is said to be strongly solvable in $H_{p, \theta}$ if, for any $a > 0$, the condition $f \in H_{p, \theta}(0, a)$ implies the existence of a unique solution to Eq. (5).

Definition 6. A pair of vector functions (v, f) defined in $[0, a)$ is called a weak solution to the system

$$v + B(Tv, Tv) = f,$$
$$g + (B_{Tv}T)^* g = 0, \quad g = |A^\theta f|_H^{p-2} A^{2\theta} f \tag{6}$$

if the following conditions are satisfied:

(a) $f(\cdot) \in H_{p, \theta}(0, a)$ and $v \in H_{p, \theta}(0, a - \varepsilon)$ for any $\varepsilon > 0$.

(b) If $\varepsilon \in (0, a)$, then the first equation holds on $(0, a - \varepsilon]$ in $H_{p, \theta}(0, a - \varepsilon)$ in the conventional sense.

(c) If $\varepsilon \in (0, a)$ and $m(t)$ is a vector function from $C^\infty(H; 0, a)$ that vanishes on $[a - \varepsilon, a]$, then the integral of the function $\langle g(t), m(t) + B_{Tv}Tm(t) \rangle_H$ over the interval from 0 to $a - \varepsilon$ is equal to zero.

When $p = 2$ and $\theta = 0$, Definitions 1 and 5 give the same meaning of strong solvability (if we set $v = u' + Au$ in (5)). Moreover, the analogue of Theorem A is the following result.

Theorem C. *Suppose that conditions* (2) *hold for all* γ_0 *and* γ *obeying* $2\gamma_0 = 2\delta - \gamma$ *and* $-\infty < 4\gamma \le 3$, $\delta > 0$, *where* δ *is a positive constant, and let c in* (2) *depend on* γ. *Assume that* θ *and q satisfy the inequalities*

$$0 \le q(2\delta - \theta) < 2, \quad 0 \le 2\delta - \theta < 1, \quad 1 < q < \infty,$$

$$p^{-1} + q^{-1} = 1.$$

Equation (5) *is strongly solvable in* $H_{p,\theta}$ *if and only if, for any* $a > 0$, *the* (*weak*) *solution to system* (6) *is the trivial solution.*

If Eq. (5) is strongly solvable in $H_{p,\theta}$, then the solution v changes little when f is slightly perturbed in the $H_{p,\theta}$ norm. Moreover, if f depends analytically on a parameter ξ, then v is also an analytic function of this parameter. Thus, in view of the strong solvability, perturbation theory can be used in full. This is a factor motivating the high interest in the existence of a global strong solution to the Navier–Stokes equations.

By using standard techniques of perturbation theory, we can deduce the following result from Theorem C.

Theorem D. *Let conditions* (2) *be satisfied for all* γ_0 *and* γ *obeying* $2\gamma_0 = 2\delta - \gamma$ *and* $-\infty < 4 \le 3$, *where* δ *is a constant. Suppose that c in* (2) *depends on* γ. *It is assumed that, for some* θ *and q satisfying the inequalities*

$$0 \le q(2\delta - \theta) < 2, \quad 0 \le 2\delta - \theta < 1, \quad 1 < q < \infty,$$

Eq. (5) *is strongly solvable in* $H_{p,\theta}$, *where* $p^{-1} + q^{-1} = 1$. *Then Eq.* (5) *is strongly solvable in* $H_{r,\beta}$ *if*

$$0 \le s(2\delta - \beta) < 2, \quad 0 \le 2\delta - \beta < 1, \quad s^{-1} + r^{-1} = 1.$$

Now consider the Navier–Stokes equations

$$\frac{\partial}{\partial t}u_j - \Delta u_j + \frac{\partial}{\partial x_l}u_l u_j + \frac{\partial}{\partial x_j}P = f_j, \quad j = 1, 2, 3,$$
$$\operatorname{div} u = \frac{\partial}{\partial x_k}u_k = 0 \qquad (7)$$

in the domain $Q \times [0, a]$, where $a > 0$ and $Q = \{x: -\pi < x_j < \pi, j = 1, 2, 3\}$. System (7) is supplemented with the zero initial conditions with respect to time and periodic boundary conditions with respect to spatial variables for

$$u = (u_1, u_2, u_3), \ P, \ \frac{\partial}{\partial x_k}u, \ \frac{\partial}{\partial x_k}f, \quad k = 1, 2, 3. \quad (8)$$

The pressure P in (7) can be determined up to a time-dependent function. For P to be uniquely determined, we add the condition $P_{av}(t) = 0$ for all $t > 0$, where P_{av} denotes the average over Q.

Applying the preceding results, we derive the following statement.

Theorem E. *Problem* (7), (8) *with an arbitrary* $f = (f_1, f_2, f_3)$ *satisfying* $f_j \in L_2(Q \times (0, a))$ $(j = 1, 2, 3)$ *has a solution* $(u, P) = (u_1, u_2, u_3; P)$ *obeying* $u' - \Delta u$, $\operatorname{grad} P \in L_2(Q \times (0, a))$ *if and only if the system of equations*

$$\frac{\partial}{\partial t}u_j - \Delta u_j + \frac{\partial}{\partial x_l}u_l u_j + \frac{\partial}{\partial x_j}P = f_j,$$
$$\frac{\partial}{\partial x_k}u_k \equiv \operatorname{div} u = 0,$$
$$-\frac{\partial}{\partial t}f_j - \Delta f_j - f_{l x_j}u_l - f_{j x_l}u_l + \frac{\partial}{\partial x_j}r = 0, \qquad (9)$$
$$\frac{\partial}{\partial x_k}f_k \equiv \operatorname{div} f = 0, \quad j = 1, 2, 3,$$
$$P_{av} = r_{av} = 0$$

with the Cauchy conditions $u_j(0) = 0$ *and* $f_j(a) = 0$ $(j = 1, 2, 3)$ *and with periodic boundary conditions in spatial variables for*

$$u = (u_1, u_2, u_3), f = (f_1, f_2, f_3),$$
$$P, \ r, \ \frac{\partial}{\partial x_k}u, \ \frac{\partial}{\partial x_k}f, \ k = 1, 2, 3$$

has only the trivial solution (*see Definition* 3 *of a weak solution*).

This theorem is a consequence of Theorem A.

If the boundary conditions are not periodic and, say, the no-slip conditions are required to hold, then the transition to an abstract statement is complicated. The difficulties are associated with the fact that the resolvent of the Laplacian does not commute with the differentiation operations. Nevertheless, this transition is possible if we use the results of [1, 2].

It is easy to show that the classical or strong solution to problem (9) is only the trivial solution. However, Theorem E deals with a weak solution in the sense of Definition 3. Therefore, Theorem E does not solve the problem of the existence of a global strong solution to the Navier–Stokes equations but rather gives its equivalent restatement.

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